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ON GENERATING FUNCTIONS OF POLYNOMIAL SYSTEMS

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1. Generating functions and generating sequences. If the function $g(x, t)$ admits a formal expansion in powers of t in which the coefficient of $t^n/n!$ is a polynomial in x, $y_n(x)$, of degree n for $n=0, 1, 2, \cdots$, it readily follows that $g(x, t)$ may be written in the form

$$g(x, t) = \sum y_n(x)t^n/n! = \sum \phi_n(t)(xt)^n/n!, \tag{1}$$

where $\phi_n(t)$ admits a formal expansion in powers of t; to insure the existence of a complete sequence of polynomials $y_n(x)$ we must have $\phi_n(0)\neq 0$ for all n. All indicated summations have the range 0 to ∞.

We shall call $g(x, t)$ the *generating function* (g.f.) and $[\phi_n(t)]$ the *generating sequence* (g.s.) of the sequence of polynomials $[y_n(x)]$.

By successive differentiation of (1) with respect to x we obtain

$$t^{-k}D_x^{(k)}g(x, t) = \sum \phi_{n+k}(t)(xt)^n/n! = \sum \frac{y_{n+k}^{(k)}(x)}{(n+1)\cdots(n+k)}\cdot\frac{t^n}{n!}, \tag{2}$$

where the superscripts (k) indicate orders of derivatives. We have here the g.f. and the g.s. of the sequence of polynomials $[y_{n+k}^{(k)}/(n+1)\cdots(n+k)]$, $k=1, 2, \cdots$.

Differentiation of (1) with respect to t and reduction by use of (2) for $k=1$ gives

$$\sum \phi_n'(t)(xt)^n/n! = \sum [y_{n+1} - xy_{n+1}'/(n+1)]t^n/n!, \tag{3}$$

and by iteration, with

$$y_{n,1} \equiv y_{n+1} - xy_{n+1}'/(n+1) \quad \text{and} \quad y_{n,r+1} \equiv y_{n+1,r} - xy_{n+1,r}'/(n+1),$$

$$\sum \phi_n^{(r)}(t)(xt)^n/n! = \sum y_{n,r}t^n/n!. \tag{4}$$

That is, the sequence of rth derivatives of the terms of $[\phi_n]$ is the g.s. of $[y_{n,r}]$. To make the last sequence complete we assume $\phi_n^{(r)}(0)\neq 0$ for all n and r.

2. Appell polynomials. If, for all n,

$$\phi_n(t) = \phi(t) = \sum a_n t^n/n!, \qquad a_0 \neq 0,$$

we have

$$g(x, t) = \phi(t)e^{xt} = \sum A_n(x)t^n/n!, \tag{5}$$

where $A_n(x)$ is the nth Appell polynomial [1]. These polynomials were introduced by Appell as solutions of the defining equation

$$D_x A_n(x) = nA_{n-1}(x), \tag{6}$$

which is a generalization of $D_x x^n = nx^{n-1}$, and have received considerable atten-

tion in the literature. They may be written conveniently in the form of the symbolic binomial expansion

$$A_n(x) = (a + x)^{(n)}, \tag{7}$$

where, after expanding, we replace a^r by a_r from the expansion of $\phi(t)$.

Equation (6) follows readily from (5) by use of (2) with $k=1$.

It is obvious that, when $\phi_n(t)$ is independent of n, the $y_{n,r}$ in (4) are Appell polynomials for all values of n and of r.

3. The polynomials $X_n(x)$ and $Y_n(x)$. If, in (1), we take $\phi_n(t)=b_n\phi(t)$, $b_n\neq 0$ for all n, and designate the resulting polynomials by $X_n(x)$, we obtain

$$g(x, t) = \sum b_n\phi(t)(xt)^n/n! = \sum X_n(x)t^n/n!. \tag{8}$$

As before, let

$$\phi(t) = \sum a_n t^n/n! \tag{9}$$

and also let

$$f(xt) = \sum b_n(xt)^n/n!. \tag{10}$$

Then

$$g(x, t) = \phi(t)f(xt) = \sum X_n(x)t^n/n!, \tag{11}$$

and the $X_n(x)$ may be written in the symbolic form

$$X_n(x) = (a + bx)^{(n)}, \tag{12}$$

where, after expanding, we replace a^r, b^r by a_r, b_r respectively.

The Appell polynomials are a subclass of the $X_n(x)$'s for the case $b_n=1$, all n.

Another interesting subclass of (12) results when we take $a_n=1$, all n. We shall designate this subclass by $Y_n(x)$. Then

$$g(x, t) = e^t f(xt) = \sum Y_n(x)t^n/n!; \tag{13}$$

$$Y_n(x) = (1 + bx)^{(n)} \quad \text{and} \quad f(xt) = \sum b_n(xt)^n/n!. \tag{14}$$

When necessary to indicate the dependence of $Y_n(x)$ on the b's, we use $Y_n(x, b_r)$.

Equation (3), with $\phi_n(t)=\phi_n'(t)=b_ne^t$, gives the fundamental relation

$$xY_n' = n(Y_n - Y_{n-1}) = n\Delta Y_{n-1}. \tag{15}$$

By iteration there is indicated the relation

$$x^k D_x^{(k)} Y_n = \frac{n!}{(n-k)!}\Delta^k Y_{n-k}, \tag{16}$$

which is readily proved by induction.

Every $g(x, t)$ of the form (13) satisfies the partial differential equation

$$tg_t - xg_x = tg, \tag{17}$$

where subscripts indicate partial derivatives. From this equation and (13) we again obtain (15).

By use of well known properties of the binomial coefficients we find the relation

$$b_n x^n = (Y - 1)^{(n)}; \qquad Y^{(r)} \equiv Y_r = Y_r(x, b_s). \tag{18}$$

By means of this relation a polynomial in x, of degree n, can be expressed as a linear combination of $Y_0, Y_1, \cdots, Y_n$.

$$\sum_{r=0}^{n} b_r x^r = \sum_{r=0}^{n} A_{n,r} Y_r(x, b_s); \qquad A_{n,r} = \sum_{i=0}^{n-r} (-1)^i \binom{r+i}{r}.$$

To obtain examples of $Y_n(x)$ we choose for $f(xt)$ some simple power series without missing terms and indicate the values of b_n from which the polynomials may be written down explicitly by use of (14).

(A) $f(xt) = \cos (xt)^{1/2}$, $\qquad b_n = (-1)^n n!/(2n)!$.

(B) $f(xt) = \sin (xt)^{1/2}/(xt)^{1/2}$, $\qquad b_n = (-1)^n n!/(2n+1)!$.

(C) $f(xt) = \cosh (xt)^{1/2}$, $\qquad b_n = n!/(2n)!$.

(D) $f(xt) = J_0(2(xt)^{1/2})$, $\qquad b_n = (-1)^n/n!$.

These functions satisfy a simple type of linear differential equation in x of second order, namely,

$$xf_{xx} + \alpha f_x + \beta t f = 0; \qquad \alpha, \beta \text{ constants.} \tag{19}$$

Obviously the associated $g(x, t)$ also satisfies (19). By use of (1), (15) and (19) we can obtain the following differential equation and difference equation for the corresponding $Y_n(x)$.

$$xY_n'' + (\alpha - \beta x)Y_n' + n\beta Y_n = 0. \tag{20}$$

$$(n - 1 + \alpha)Y_n = (2n - 2 + \alpha - \beta x)Y_{n-1} - (n-1)Y_{n-2}. \tag{21}$$

To get (20) and (21) we substitute from (1) in (19) and compare coefficients of t^n, obtaining

$$xY_n' + \alpha Y_n' + n\beta Y_{n-1} = 0. \tag{a}$$

Equation (15) is

$$xY_n' = nY_n - nY_{n-1}. \tag{b}$$

Replacing n by $(n-1)$ in (b) gives

$$xY_{n-1}' = (n-1)Y_{n-1} - (n-1)Y_{n-2}. \tag{c}$$

Differentiation of (b) gives

$$xY_n'' = (n-1)Y_n' - nY_{n-1}'. \tag{d}$$

Suitable combinations of these four equations give (20) and (21).

For the four examples considered above the values (α, β) are, respectively, $(1/2, 1/4)$, $(3/2, 1/4)$, $(1/2, -1/4)$, $(1, 1)$.

We define the systems of polynomials $y_n(x)$ and $c_n y_n(a+bx)$ to be *equivalent*, where c_n does not depend on x. Accordingly, (20) is the same for all values of c_n, but (21) and the function $g(x, t)$ will depend essentially on the choice of c_n.

4. Orthogonal polynomials, (*OP*). We use the standard notation

$$(u, v) = (u, v; a, b; \theta) = \int_a^b u(x)v(x)d\theta(x), \tag{22}$$

and, when $\theta(x)$ is the integral of $w(x)$,

$$(u, v) = (u, v; a, b; w) = \int_a^b u(x)v(x)w(x)dx, \tag{22'}$$

where $w(x)$ is called the weight function.

Associated with each function $\theta(x)$ or $w(x)$ is a sequence of moments [2],

$$u_n = \int_a^b x^n d\theta = (x^n, 1) = (x^n, 1; a, b; \theta),$$

and similarly with w in place of θ.

If $y_n(x)$, $n=0, 1, 2, \cdots$, is an *OP* system with respect to a given interval and a given $\theta(x)$ or $w(x)$, we have, by definition,

$$(y_n, y_m) = (y_n, y_m; a, b; \theta \text{ or } w) = 0, \qquad n \neq m.$$

We use the further notation

$$(x^r, y_n) = n!k_{r,n} \quad \text{with} \quad k_{r,0} = u_r; \qquad (y_n, y_n) = n!l_n.$$

If

$$x^r = \sum_{n=0}^{r} c_{r,n} y_n(x),$$

then $c_{r,n}l_n = k_{r,n}$; $r \geqq n$. For $n > r$, $k_{r,n} = 0$, and $c_{r,n}$ is taken equal to zero by definition.

By substituting the values of $g(x, t)$ given in (1) in the integral (x^r, g) we obtain formally, for all r,

$$\sum u_{n+r}t^n\phi_n(t)/n! = \sum k_{r,n}t^n = \sum c_{r,n}l_n t^n. \tag{23}$$

If $\phi_n(t)$ is given by

$$\phi_n(t) = \sum a_{n,r}t^r, \qquad a_{n,0} \neq 0 \text{ for all } n,$$

equations (23) set up necessary relations between the $a_{n,r}$ and $k_{r,n}$. For the polynomials $Y_n(x, b_s)$ of (13) these equations become

$$b_n u_{r+n} = \sum_{i=0}^{r} (-1)^{n-i} \binom{n}{i} i! k_{r,i}.$$

When $r=0$, we have, on taking $u_0 = k_{0,0} = 1$,

$$b_n u_n = (-1)^n; \qquad n = 0, 1, 2, \cdots.$$

Therefore a necessary condition that the $Y_n(x, b_s)$ form an orthogonal system is that the reciprocals of the quantities $(-1)^n b_n$ shall form a moment sequence.

5. Orthogonal polynomials among the $Y_n(x)$. It has been shown by several writers that among the Appell polynomials the only *OP* are those of Hermite. That there are also *OP* among the $Y_n(x)$ follows immediately from the formula [3]

$$g(x, t) = e^t (xt)^{-\alpha/2} J_\alpha[2(xt)^{1/2}] = \sum L_n^{(\alpha)}(x) t^n / \Gamma(n + \alpha + 1).$$

On comparing the known equations [4] for $L_n^{(\alpha)}(x)$, namely,

$$xD_x L_n^{(\alpha)} = nL_n^{(\alpha)} - (n + \alpha) L_{n-1}^{(\alpha)},$$

$$xD_{xx} L_n^{(\alpha)} + (\alpha + 1 - x) D_x L_n^{(\alpha)} + nL_n^{(\alpha)} = 0,$$

with our corresponding equations (15) and (20), we find that the polynomials determined by (19) are the generalized Laguerre polynomials:

$$Y_n(x) \equiv Y_n^{(\alpha,\beta)}(x) = n! L_n^{(\alpha-1)}(\beta x)/\Gamma(n + \alpha). \tag{24}$$

In the four examples previously considered, the equations (20) and (21) show that the $Y_n(x)$ in (*A*), (*B*) and (*D*) are *OP*, and also they that are special cases of (24) or equivalent systems. The question whether there are among the $Y_n(x)$ other *OP* than the generalized Laguerre polynomials (24) remains open.

Example (*C*) does not lead to *OP*. Thus there is at least a suggestion that, if we are to obtain *OP* from (19), the choice of α and β should be restricted to values which will make the differential equation (20) oscillatory.

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Algebraic or Transcendental? Let $a_k = 0$ or 1, according as the kth decimal in π is less than 5, or greater than or equal to 5. Is $\sum_1 a_k 2^{-k}$ algebraic or transcendental? The same question can be asked for any other real number.

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WHAT IS A TOPOLOGICAL GROUP?

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1. Introduction. The concept of a topological group originated with Lie, who was concerned with a very important special class of these groups now known as Lie groups. Much of the work of Lie and of many of his followers was on the theory in the small which is a way of saying that they studied only a small part of the group near the identity element. After a theory of topological spaces had been developed it was possible to formulate Lie's idea in somewhat more abstract terms and thus obtain topological groups in full generality. Progress in topology had also provided tools and a language which made it easier to discuss these groups in the large, and it was inevitable that a greater interest in questions in the large should arise. This is not a denial of the fruitfulness of the local point of view, which with the related subject of Lie algebras, is an indispensable part of the theory.

2. Groups and spaces. A collection of elements is called a group provided that there is given for each pair of elements g_1 and g_2 an operation called the product, written as g_1g_2, satisfying the following axioms:

A. g_1g_2 is in G.

B. $g_1(g_2g_3)=(g_1g_2)g_3$.

C. There is an identity element e such that for any g in G, $ge=eg=g$.

D. For each g in G there is an inverse element g^{-1} such that $gg^{-1}=g^{-1}g=e$.

In this set of axioms as in others to follow, no attempt is made to present the minimum possible conditions.

Topological groups are spaces and to understand them it is necessary to have some notion of what a space is. Almost any idea of a space the reader may have will be adequate to find out something about topological groups. One thing which might be meant by a space is that it is any subset of a euclidean space of some number of dimensions. This would be slightly restrictive but would include many of the cases of interest. For the sake of completeness the axioms for a topological space, as now usually given, are listed below, but the chief thing to have in mind is an intuitive concept of continuous transformations and functions.

A collection G of elements is called a topological space provided a certain family of subsets called open sets is given so as to satisfy the following conditions:

A. The null set and the entire space are open sets.

B. The union of any number of open sets is an open set.

C. The intersection of a finite number of open sets is an open set.

In addition, for topological groups it is usually convenient to assume the following separation property and it is assumed here.

D. If g_1 and g_2 are two points of G, then there is an open set which includes g_1 and does not include g_2.

3. Topological groups. A collection G of elements is called a topological group provided the collection satisfies the following axioms:

A. G is a group.

B. G is a topological space.

C. The group operations g_1g_2 and g^{-1} are continuous functions in the topology of the space.

The last condition is the link between the space and group properties and without it the group and space properties would be entirely independent of each other.

Example 1. The real numbers form both a space and a group where the group "product" is defined to be the sum. The group operations are continuous in the usual topology of the line, and consequently the real numbers form a topological group.

Example 2. The complex numbers can be represented by the points in the plane. They form a group under addition, and the group operations are continuous in the topology of the plane. Thus we obtain another topological group.

Incidentally the plane can be made into a topological group in another and essentially different way as follows: If (x, y) and (x', y') are two points of the plane the product of these two points is defined to be the point with coordinates $(x+x'e^{-y},\ y+y')$. That this rule of combination really yields a topological group is left as an easy exercise for the reader. The group multiplication defined in this way is noncommutative so that this group is essentially different from the group of the complex numbers under addition although the underlying spaces in the two cases are the same.

Example 3. Let G be the collection of all complex numbers of absolute value one. This collection G forms a topological group if the rule of combination is ordinary multiplication of complex numbers. The underlying space is a circumference of unit radius. The group is isomorphic to the group of real numbers modulo 1. This group is an important one not only for the study of other topological groups but also because of applications in combinatorial topology.

Example 4. Let G be the collection of all n by n real matrices with nonvanishing determinant. These form a group under multiplication. Any matrix of this kind can be regarded as a point in n^2 dimensional space simply by regarding the elements of the first row as the first n coordinates, the elements of the second row as the next n coordinates, and so on. Hence any collection of n by n matrices forms a topological space because it may be regarded as a subset of euclidean space. Since each element of a product matrix is a polynomial in the elements of the two matrices being multiplied it is quite clear that the group product, is continuous. The group inverse can be expressed quite simply and is also continuous, and therefore the collection G is a topological group. Of course entirely analogous remarks could be made for n by n complex matrices with nonvanishing determinant.

Any subgroup of a topological group is also a topological group, and this shows that any of the well known groups of matrices are topological groups. In

particular this remark applies to the group of orthogonal matrices. From the algebraic condition for such matrices it can be seen that the associated point set in n^2 dimensional euclidean space is closed and bounded. Consequently the underlying space of the orthogonal group is compact, for a compact topological space is merely one which is analogous to a closed bounded set. This is also true for the group of unitary matrices.

Example 5. Let $C_1, C_2, C_3, \cdots$ be an infinite number of copies of the circle group, that is the group given in Example 3. The elements of any C_i may be given by means of an angle. Let $f_1, f_2, \cdots$ be a sequence of homomorphisms where f_i carries the group C_{i+1} onto the group C_i by doubling the angle of each element. This is a two to one homomorphism. Let G be the collection of elements each of which is an infinite sequence

$$(c_1, c_2, c_3, \cdots) \tag{1}$$

where $c_i = f_i(c_{i+1})$. If

$$(c_1', c_2', c_3', \cdots)$$

is any other element of G, the product of these two elements is defined to be

$$(c_1c_1', c_2c_2', c_3c_3', \cdots).$$

That this is an element of G and that this rule of multiplication satisfies the other axioms for a group can be verified quite easily. Hence the collection G forms a group.

Let i be any integer and let O_i be any open subset of C_i. Then let O^* be the collection of all elements (1) having the property that c_i is in O_i. By definition O^* will be called an open set in G. The topology of G is to be specified by choosing for the open sets all sets which may be defined in the above described manner. It can be shown that under this definition G becomes a topological space and that the group operations are continuous in this topology. The details are omitted.

The space obtained in this way is compact and one dimensional but it has a rather complicated structure; it has been called a solenoid by van Dantzig.

4. Properties of group spaces. If a is a fixed element of a topological group G and if x is a variable element then the mapping

$$T_a: \quad x \to ax$$

is a homeomorphism of G onto itself. If b and c are any two elements of G and if we choose $a = cb^{-1}$ then T_a maps the point b on the point c. We see therefore that for any two elements b and c of G there is a homeomorphism of G onto itself taking b to c. This means that neighborhoods of the points b and c are alike topologically, and this implies a drastic restriction on spaces which can be carriers of topological groups. A closed interval can not be made into a topological group, for example, because an end point and an interior point of the interval

do not have topologically equivalent neighborhoods. In the same way many other spaces can be eliminated as carriers of a topological group.

If a is not the identity then the mapping T_a does not have a fixed point. If G is arcwise connected then T_a can be shown to be a deformation. There are many spaces which do not admit a deformation without a fixed point. We can see in this way that no even dimensional sphere can be the carrier of a topological group. It can be shown in other ways that there are only two spheres which can carry a topological group, namely the one dimensional sphere (example 3) and the three dimensional sphere. The reader familiar with quaternions may see that the three dimensional sphere is a topological group by observing that the quaternions of absolute value one form a group under multiplication and that those same quaternions are the points on the surface of the unit sphere in four space.

Many other properties of group spaces are known and from the ones mentioned here and others it can be verified that the only connected and well behaved (manifolds; see next section) one and two dimensional spaces which can carry a group are the circle, line, plane, cylinder, and torus.

5. Lie groups. The groups in the first four examples above are Lie groups and the one in the fifth example is not a Lie group. In order to define Lie groups it will be necessary to define manifolds. A manifold in the sense used here is a topological space such that every point is in an open set homeomorphic to the set $x_1^2+x_2^2+\cdots+x_n^2<1$ in euclidean n-space. For example, the circle and line are one dimensional manifolds, the torus, plane and cylinder are two dimensional manifolds, and so on. In a manifold we may clearly choose a coordinate system valid for some (possibly small) open set including each point. If there is given a set of coordinate systems which taken together cover the entire manifold and if whenever two systems overlap the transformation from one to the other is analytic then the manifold is called an analytic manifold.

A Lie group is a topological group whose underlying space is an analytic manifold and in which the group operations are specified by means of analytic functions of the coordinates.

If e is the identity element then e will be in a subset O for which there is given a definite homeomorphism with $x_1^2+\cdots+x_n^2<1$. This gives a definite set of coordinates for each point of O. If x, y, and xy (if x and y are near enough to e and xy must be in O because of the continuity of the product) are points of O, then the ith coordinate of xy is given by a function of the coordinates of x and y

$$(xy)_i = f_i(x_1, \cdots, x_n; y_1, \cdots, y_n).$$

According to the definition of a Lie group this function must be analytic in the $2n$ variables, that is it can be expanded in a power series in these variables. This small open set O was the entity mainly studied by Lie and his followers. Such a small open set O is often called a group germ. The group germ yields much information about the group but it clearly can not give complete information be-

cause there are groups which are distinct in the large and yet have identical group germs, for example the line and circle as given in Examples 1 and 3.

For the study of the group operations in O, we may use a great many results from analysis because of the analyticity of the functions f_i. It is this fact which simplifies the theory of Lie groups and makes it possible to obtain far more information in this case than in any other.

As we have already remarked, all the groups mentioned in the first four examples are Lie groups. This is fairly clear for the groups of the first three examples. For groups of matrices it is not quite so obvious although it is not very difficult. The underlying space of the group of Example 5 is not a manifold and therefore this group can not be a Lie group.

6. Measure in groups. If the group G is locally compact then Haar has shown how a measure can be defined in G which is invariant under left translations. The measure, like any measure, is a generalization of volume and is a real non-negative function defined on certain sets, in particular on all open and closed sets. It has various useful properties which make it entirely comparable to ordinary Lebesgue measure in euclidean spaces. By means of this measure, a theory of integration can be developed which enjoys the properties of the ordinary Lebesgue integral. These integrals have been of great use in many questions related to groups, particularly in studying the representations of topological groups by means of orthogonal and unitary matrices.

It is a consequence of the results thus developed that every compact topological group can be approximated by means of compact Lie groups somewhat as the group of Example 5 is approximated by means of the circle groups used in defining it.

7. Hilbert's fifth problem. Some topological groups fail to be Lie groups because their underlying spaces are not manifolds. But if a space is a manifold can it fail to be a Lie group? In other words, if a topological group G is a manifold, can a system of coordinates always be introduced into G in such a way that G becomes a Lie group? This question has been answered affirmatively for the compact case by von Neumann, for the Abelian case by Pontrjagin, and for the solvable case by Chevalley. The general case, however, remains open and the methods so far used do not extend to it. One method of attack has been to try to show that sufficiently small neighborhoods of the identity can contain no subgroups except the trivial one containing only the identity. This might be a partial step toward the solution of the problem, but so far attempts to prove this have also failed. This problem is a part of the famous fifth problem of Hilbert which is a very important unsolved problem.

8. Transformation groups. If each element g of a topological group G is a homeomorphism $g(x)$ of a space M onto itself and if these homeomorphisms combine in the usual way and depend continuously on G and M simultaneously, then G is called a transformation group of M. Furthermore if all these homeo-

morphisms are distinct the group is called effective. If G and M are both manifolds then Hilbert asked in his fifth problem whether coordinates in G and M may be so chosen that all operations are analytic in the parameters of G and M simultaneously. J. von Neumann has pointed out that this can not always be done if G is not compact. Whether or not this can always be done in the compact case is unknown. When G is compact and transitive on M it can always be done and this follows quite easily from the result of von Neumann already mentioned.

Hilbert's conjecture has been extended a bit by several people in a way which we shall now state. Let M be a manifold and let G be a compact effective transformation group of M. Then the extended conjecture is that G itself must be a manifold, but this also is unverified. If it were known to be true then we could return to Hilbert's question above and inquire whether or not analytic parameters could be introduced.

Combining these various statements we may sum up the spirit of Hilbert's conjecture on transformation groups for the compact case as follows; If G is a compact effective transformation group of a manifold M then the structure and action of G must be comparatively simple. With this simple statement before us it becomes clear that we are dealing with a profound and fundamental question which lies at the basis of geometry, and this was the point of view of Hilbert. Certain progess in the case where M is 3 dimensional has been made by Leo Zippin and the author. A very important case arises when G is finite, and in this case P. A. Smith has many beautiful theorems which give considerable support to the conjecture. Newman has an important paper on this case which points in the same direction.

Even granting that the compact effective group G acts analytically on a manifold M, the topology of the resulting situation in the large has been explored but little, and is a very interesting field for investigation. It appears likely that these and related questions will provide topology with a rich source of problems for some time to come.

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POLYNOMIALS WHOSE ZEROS HAVE NEGATIVE REAL PARTS*

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1. Introduction. In the problem of stability of dynamical systems it is important to have simple conditions that a polynomial with real coefficients shall have only zeros with negative real parts. Although such conditions are known, it has seemed desirable to formulate and establish them in a simple way, and to make them more widely available. We have obtained the following theorem.

THEOREM A. *Let*

$$P(z) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n \tag{1.1}$$

be a polynomial with real coefficients, and let

$$Q(z) = a_1 z^{n-1} + a_3 z^{n-3} + a_5 z^{n-5} + \cdots \tag{1.2}$$

be the polynomial obtained from $P(z)$ by dropping out the first, third, fifth, $\cdots$ terms. Then all the zeros of $P(z)$ have negative real parts if and only if

$$\frac{Q(z)}{P(z)} = \cfrac{1}{c_1 z + 1 + \cfrac{1}{c_2 z + \cfrac{1}{c_3 z + \ddots + \cfrac{1}{c_n z}}}}, \tag{1.3}$$

where the coefficients $c_1, c_2, \cdots, c_n$ are all positive.

Thus, to test a given polynomial $P(z)$, one applies the euclidean algorithm for the greatest common divisor, to the polynomials $P(z)$ and $Q(z)$. If the successive quotients are of the form $c_1 z+1$, $c_2 z$, $\cdots$, $c_n z$, where the c_p are positive, then the zeros of $P(z)$ all have negative real parts. In the contrary event, there must be at least one zero with real part greater than or equal to zero. The proof of Theorem A is contained in §2.

In §3 we have given necessary and sufficient conditions for a polynomial (1.1) with complex coefficients to be the last denominator of a continued fraction (1.3), namely the conditions that all the determinants

$$D_1 = a_1,\ D_2 = \begin{vmatrix} a_1, & a_3 \\ 1, & a_2 \end{vmatrix}, \cdots, D_n = \begin{vmatrix} a_1, & a_3, & \cdots, & a_{2n-1} \\ 1, & a_2, & \cdots, & a_{2n-2} \\ 0, & a_1, & \cdots, & a_{2n-3} \\ \cdot & \cdot & \cdots & \cdot \\ & & & a_n \end{vmatrix}, \tag{1.4}$$

* Presented to the American Mathematical Society, April, 1945. This paper was written as the result of a conversation with Professor H. S. Snyder of the Department of Physics, Northwestern University.

($a_0=1$, $a_p=0$ for $p<0$ or $p>n$) shall be different from zero. By means of a known theorem on continued fractions we then give an upper bound for the moduli of the zeros of any polynomial $P(z)$ for which (1.3) holds, (§4).

In §5 we have shown that if $P(z)$ has real coefficients, and (1.3) holds with $c_p\neq 0$, $p=1, 2, \cdots, n$, then the number of zeros of $P(z)$ with positive real parts is equal to the number of negative terms in the sequence $c_1, c_2, \cdots, c_n$, and the number of zeros with negative real parts is equal to the number of positive terms in this sequence.

There is a numerical example at the end of §5.

Finally, we have given in §6 a brief survey of the literature of these and related problems.

2. Proof of Theorem A. We suppose first that (1.3) holds, where the coefficients c_p are all positive, and shall prove that all the zeros of $P(z)$ have negative real parts. It will be convenient to make the substitution

$$b_0 = 1/c_1, \qquad b_p = 1/c_p c_{p+1}, \qquad p = 1, 2, \cdots, n-1, \tag{2.1}$$

so that the continued fraction takes the form

$$\frac{Q(z)}{P(z)} = \cfrac{b_0}{z + b_0 + \cfrac{b_1}{z + \cfrac{b_2}{z + \cfrac{\ddots}{\ddots + \cfrac{b_{n-1}}{z}}}}}. \tag{2.2}$$

This has the advantage that the coefficients of the highest powers of z in the successive denominators $P_1(z)=z+b_0$, $P_2(z)=z^2+b_0z+b_1$, $\cdots$, $P_n(z)\equiv P(z)$, are equal to unity. The b_p are evidently positive if and only if the c_p are positive. If we let $Q_1(z)=b_0$, $Q_2(z)=b_0z$, $\cdots$, $Q_n(z)=Q(z)$ be the successive numerators, then we have the determinant formula

$$Q_m(z)P_{m-1}(z) - Q_{m-1}(z)P_m(z) = (-1)^{m-1}b_0b_1\cdots b_{m-1}, \tag{2.3}$$

$m=1, 2, \cdots, n$, ($Q_0=0$, $P_0=1$). If we put $m=n$ in this formula we conclude at once that $Q(z)/P(z)$ is irreducible.

Consider the linear fractional transformations*

$$t = \frac{b_0}{z + b_0 + w_1}, \quad w_1 = \frac{b_1}{z + w_2}, \quad \cdots, \quad w_{n-1} = \frac{b_{n-1}}{z + w_n},$$

in the variables t, w_1, w_2, $\cdots$, w_n, z being a parameter with nonnegative real part. The value of the rational fraction $Q(z)/P(z)$ is the image of the point

* This argument was used, for instance, in [2], pp. 105–107.

$w_n = 0$ under the product of these transformations. Since, for $R(w_1) \geqq 0$, $R(z) \geqq 0$, we have

$$|t - \tfrac{1}{2}| \leqq \tfrac{1}{2}, \tag{2.4}$$

and since $R(w_p) \geqq 0$ implies $R(w_{p-1}) \geqq 0$, $p = 2, 3, \cdots, n$, it follows that if $R(w_n) \geqq 0$, then t satisfies (2.4). Hence, in particular,

$$\left| \frac{Q(z)}{P(z)} - \tfrac{1}{2} \right| \leqq \tfrac{1}{2}, \quad \text{if} \quad R(z) \geqq 0. \tag{2.5}$$

Thus, $Q(z)/P(z)$ is bounded for $R(z) \geqq 0$, and, since it is irreducible, the denominator $P(z) \neq 0$ for $R(z) \geqq 0$.

We now suppose, conversely, that $P(z)$ is an arbitrary polynomial of the form (1.1) with real coefficients, whose zeros all have negative real parts. If we regard $|P(z)|$ as the product of the lengths of the vectors from z to its zeros, we then see immediately that $|P(z)| \geqq |P(-z)|$ for $R(z) \geqq 0$, where inequality holds in both if it holds in either. Therefore,

$$|P(z) \pm P(-z)| \geqq |P(z)| - |P(-z)| > 0 \quad \text{for} \quad R(z) > 0. \tag{2.6}$$

Similarly, we have

$$|P(z) \mp P(-z)| \geqq |P(-z)| - |P(z)| > 0 \quad \text{for} \quad R(z) < 0. \tag{2.7}$$

Let $Q(z)$ be defined as $[P(z)+P(-z)]/2$ or $[P(z)-P(-z)]/2$, according as the degree n of $P(z)$ is odd or even, respectively. Then, by (2.6), (2.7), the zeros of $Q(z)$ all lie upon the axis of imaginaries.

Moreover, since

$$\left| \frac{P(-z)}{P(z)} \right| = \left| \frac{P(-z)}{P(z)} \pm 1 \mp 1 \right| \leqq 1 \quad \text{for} \quad R(z) \geqq 0,$$

it follows that (2.5) holds.

By division we now find that

$$\frac{Q(z)}{P(z)} = \frac{1}{c_1 z + 1 + [C(z)/Q(z)]}, \qquad c_1 = 1/a_1 > 0, \tag{2.8}$$

where $C(z)/Q(z)$ is an irreducible rational fraction in which the denominator is of degree $n-1$, and the numerator is of lower degree than the denominator. Also, it follows from (2.5) and (2.8) that $R[C(z)/Q(z)] \geqq -c_1 x$, where $x = R(z) \geqq 0$. Consequently

$$R[C(z)/Q(z)] \geqq 0 \quad \text{for} \quad R(z) \geqq 0. \tag{2.9}$$

In fact, if $R[C(z)/Q(z)] = -k$, $k > 0$, for some point $z = z_0$ such that $R(z_0) > 0$, we can contradict the theorem that a nonconstant harmonic function cannot take on its minimum value on the interior of a region in which it is harmonic. It suffices to take for the region the portion of the right half-plane exterior to

circles of radius $k/2c_1$ with centers at the zeros of $Q(z)$, and interior to a circle $|z| = r > |z_0|$, so large that $|C(z)/Q(z)| \leqq k/2$ on this circle.

From (2.9) we can conclude that the zeros of $Q(z)$ are simple, and that the residues of $C(z)/Q(z)$ are positive. Otherwise, one can choose a path of z in $R(z) > 0$ approaching a pole of $C(z)/Q(z)$ in such a way that the real part of the latter will become negative. We therefore have a partial fraction development of the form

$$C(z)/Q(z) = z\sum_{p=1}^{m} \frac{2M_p}{z^2 + t_p^2}, \qquad m = [n/2], \tag{2.10}$$

where the M_p are positive, and the t_p are distinct real numbers.

We now replace z by $-iz$ in (2.10), and then multiply both members by $-i$. This gives

$$\frac{-iC(-iz)}{Q(-iz)} = \sum_{p=1}^{[n/2]} \left(\frac{M_p}{z + t_p} + \frac{M_p}{z - t_p} \right). \tag{2.11}$$

Let $M(u)$ be a step-function, constant except at the points $\pm t_p$ where it has jumps equal to M_p. Then the function (2.11) may be written as the Stieltjes integral

$$\int_{-\infty}^{+\infty} \frac{dM(u)}{z + u}.$$

Let $\sum (-1)^p k_p / z^{p+1}$ be the expansion of this function in descending powers of z, so that

$$k_p = \int_{-\infty}^{+\infty} u^p dM(u), \qquad p = 0, 1, 2, \cdots.$$

Now, the quadratic form

$$F_p(X, X) = \int_{-\infty}^{+\infty} (X_0 + X_1 u + \cdots + X_{p-1} u^{p-1})^2 dM(u)$$

is positive definite for $p = 1, 2, 3, \cdots, n-1$. Consequently, the determinants

$$\Delta_p = \begin{vmatrix} k_0, & k_1, & \cdots, & k_p \\ k_1, & k_2, & \cdots, & k_{p+1} \\ \cdots & \cdots & \cdots & \cdots \\ k_p, & k_{p+1}, & \cdots, & k_{2p} \end{vmatrix}, \qquad p = 0, 1, \cdots, n-2,$$

are all positive. Therefore, the function (2.11) has a continued fraction expansion of the form*

* *Cf.* [10] pp. 421–422 and, for instance, [11].

$$\cfrac{p_0}{q_1 + z - \cfrac{p_1}{q_2 + z - \ddots - \cfrac{p_{n-2}}{q_{n-1} + z}}},$$

where $p_0, p_1, \cdots, p_{n-2}$ are positive. Inasmuch as (2.11) is an odd function of z, it follows that $q_m = 0$, $m = 1, 2, 3, \cdots, n-1$. On replacing z by iz, and changing the notation, we then have

$$\frac{C(z)}{Q(z)} = \cfrac{1}{c_2 z + \cfrac{1}{c_3 z + \ddots + \cfrac{1}{c_n z}}},$$

where the c_p are positive. On substituting this into (2.8) we then have (1.3), and the theorem is proved.

3. Condition for a polynomial to be the denominator of a continued fraction. Let $P_n(z)$ be the nth denominator of the continued fraction

$$\cfrac{b_0}{z + b_0 + \cfrac{b_1}{z + \cfrac{b_2}{z + \ddots}}}, \qquad b_p \neq 0, \tag{3.1}$$

so that

$$P_0(z) = 1, \; P_1(z) = z + b_0, \; P_{m+1}(z) = zP_m(z) + b_m P_{m-1}(z), \quad m = 1, 2, \cdots. \tag{3.2}$$

Then $P_n(z)$ is a polynomial of degree n in which the coefficient of z^n is unity. One may show by mathematical induction that*

$$P_n(z) = z^n + b_0 z^{n-1} + S_{11} z^{n-2} + b_0 S_{21} z^{n-3} + S_{12} z^{n-4} + b_0 S_{22} z^{n-5} + \cdots, \tag{3.3}$$

where S_{ij} is the sum of all possible products of the form

$$b_{p_1} b_{p_2} \cdots b_{p_j}$$

under the condition that $i \leqq p_1 \leqq 2 + p_2 \leqq 4 + p_3 \leqq \cdots \leqq 2j - 2 + p_j \leqq n - 1$. In

* *Cf.* [7] p. 301.

order for a polynomial $P(z)=z^n+a_1z^{n-1}+a_2z^{n-2}+\cdots+a_n$ to be the nth denominator of (3.1) it is therefore necessary and sufficient that the system of nonlinear equations

$$b_0=a_1,\ S_{11}=a_2,\ b_0S_{21}=a_3,\ S_{12}=a_4,\cdots, \tag{3.4}$$

have a solution $b_0, b_1, \cdots, b_{n-1}, b_p\neq 0, p=0, 1, \cdots, n-1$.

Let us suppose first that the a_p have values such that a solution exists. Then, one may show that the determinants D_p of (1.4) are different from zero:

$$D_p\neq 0, \qquad p=1, 2, \cdots, n. \tag{3.5}$$

In fact, we have the formulas

$$\begin{aligned} D_{2k+1}&=b_0^{k+1}b_1^kb_2^kb_3^{k-1}b_4^{k-1}\cdots b_{2k-1}b_{2k}, && 2k+1\leqq n,\\ D_{2k+2}&=b_0^{k+1}b_1^{k+1}b_2^kb_3^kb_4^{k-1}\cdots b_{2k}b_{2k+1}, && 2k+2\leqq n. \end{aligned} \tag{3.6}$$

For, if $p=2k+1$ or $2k+2$, we have

$$D_p=\begin{vmatrix} b_0, & b_0S_{21}, & b_0S_{22}, & b_0S_{23}, & b_0S_{24}, \cdots\\ 1, & S_{11}, & S_{12}, & S_{13}, & S_{14}, \cdots\\ 0, & b_0, & b_0S_{21}, & b_0S_{22}, & b_0S_{23}, \cdots\\ 0, & 1, & S_{11}, & S_{12}, & S_{13}, \cdots\\ 0, & 0, & b_0, & b_0S_{21}, & b_0S_{22}, \cdots\\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

$$=b_0^{k+1}\begin{vmatrix} b_1, & b_1S_{31}, & b_1S_{32}, & b_1S_{33}, & b_1S_{34}, \cdots\\ 1, & S_{21}, & S_{22}, & S_{23}, & S_{24}, \cdots\\ 0, & b_1, & b_1S_{31}, & b_1S_{32}, & b_1S_{33}, \cdots\\ 0, & 1, & S_{21}, & S_{22}, & S_{23}, \cdots\\ 0, & 0, & b_0, & b_0S_{31}, & b_0S_{32}, \cdots\\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix},$$

as may be seen by removing the factor b_0 from the first, third, fifth, $\cdots$ rows of D_p, and then subtracting the first row from the second, the third from the fourth, the fifth from the sixth, $\cdots$. The new determinant on the right is of the same form as the original, but of order $p-1$. Therefore, the validity of (3.6) readily follows by mathematical induction. Since the b_p are different from zero, we conclude from (3.6) that (3.5) holds. Furthermore, we see that the solution of (3.4) is unique and is given by the formulas

$$b_p=\frac{D_{p-1}D_{p+1}}{D_pD_{p-1}}, \qquad p=0, 1, \cdots, n-1, \tag{3.7}$$

where we must set $D_{-2}=D_{-1}=D_0=1$.

We suppose now, conversely, that $a_1, a_2, \cdots, a_n$ are any given numbers such that (3.5) holds, and shall prove that the system (3.4) has a solution $b_p \neq 0$, $p=0, 1, \cdots, n-1$. The statement is obviously true if $n=1$. Assuming that it is true for $n-1$, $n>1$, we shall prove it for n. To that end, consider the system

$$b_1 = a_1', \ S_{21} = a_2', \ b_1 S_{31} = a_3', \ S_{22} = a_4', \cdots, \tag{3.8}$$

where $a_1', a_2', \cdots, a_{n-1}'$ are given by

$$a_1' = a_2 - (a_3/a_1), \ a_2' = a_3/a_1, \ a_3' = a_4 - (a_5/a_1), \ a_4' = a_5/a_1, \cdots.$$

Denote by D_p' the determinant D_p formed with the numbers a_p'. Then, it is easy to verify that

$$D_{2p}' = D_{2p+1}/D_1^{p+1}, \qquad D_{2p+1}' = D_{2p+2}/D_1^{p+1},$$

and consequently $D_p' \neq 0$, $p=1, 2, \cdots, n-1$. Therefore, by the hypothesis of the induction, the system (3.8) has a unique solution $b_1, b_2, \cdots, b_{n-1}$, and this is given by

$$b_{p+1} = D_{p-1}' D_{p+1}'/D_{p-1}' D_p' = D_{p-1} D_{p+2}/D_p D_{p+1}, \qquad p = 0, 1, \cdots, n-2.$$

If to these we adjoin the equation $b_0 = a_1$, it then follows that the system (3.4) has the unique solution (3.7).

We have completed the proof of the following theorem.

THEOREM B. *If $P_n(z) = z^n + a_1 z^{n-1} + \cdots + a_n$ is a polynomial of degree n whose coefficients satisfy* (3.5), *then there is determined uniquely a continued fraction of the form* (2.2) *such that $P_n(z)$ is its last denominator. The b_p are given in terms of the a_p by* (3.7). *Conversely, the coefficients in the last denominator of a continued fraction of the form* (2.2) *satisfy* (3.5).

From Theorems *A* and *B* it follows that if $P(z)$ is a polynomial of the form (1.1) with real coefficients whose zeros have negative real parts, then there exist uniquely determined polynomials $P_0=1, P_1, \cdots, P_{n-1}$, of degrees $0, 1, \cdots, n-1$, having the same properties as $P(z)$ which are connected with $P(z)$ by the recurrence formula $P_{m+1} = zP_m + b_m P_{m-1}$, $m=1, 2, \cdots, n-1$, $P_n = P$.

We note that $P(z)$ can be written in the form

$$P(z) = \begin{vmatrix} b_0+z, & -b_1, & 0, & 0, & 0, \cdots, 0 \\ 1, & z, & -b_2, & 0, & 0, \cdots, 0 \\ 0, & 1, & z, & -b_3, & 0, \cdots, 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0, & 0, & 0, & 0, \cdots, 1, & z, & -b_{n-1} \\ 0, & 0, & 0, & 0, \cdots, 0, & 1, & z \end{vmatrix}.$$

Conversely, if the b_p are positive, a polynomial of this form has all its zeros in the left half-plane, $R(z)<0$. From this expression for $P(z)$ as a determinant, it follows at once that

$$P(z) = G(z) + H(z), \tag{3.9}$$

where

$$\frac{G(z)}{H(z)} = \cfrac{b_0}{z + \cfrac{b_1}{z + \cfrac{\ddots}{\ddots \cfrac{b_{n-1}}{z}}}}. \tag{3.10}$$

We shall make use of this remark in §5.

If the b_p are positive, the fraction (3.10) has the same properties as the fraction $C(z)/Q(z)$ of (2.11). In particular, the zeros of $G(z)$ and $H(z)$ lie on the axis of imaginaries; $R[G(z)/H(z)] \geqq 0$ for $R(z) \geqq 0$; and there is a partial fraction development of the form

$$\frac{G(z)}{H(z)} = z \sum_{p=1}^{[(n+1)/2]} \frac{2M_p}{z^2 + t_p^2}, \qquad M_p > 0,\ t_1 < t_2 < t_3 < \cdots. \tag{3.11}$$

Conversely, if $G(z)/H(z)$ is an irreducible rational fraction of the form (3.11), then $P(z) = G(z) + H(z)$ has only zeros with negative real parts.

There is at least one case where the vanishing of one of the determinants D_p has a simple interpretation in terms of the zeros of the polynomial $P(z)$. We have the following theorem.

THEOREM C. *Let $P(z) = z^n + a_1 z^{n-1} + \cdots + a_n$ be a polynomial with complex coefficients. A necessary and sufficient condition for $P(z)$ and $P(-z)$ to have a zero in common is that the determinant D_n shall vanish.*

Proof. Since $a_n D_{n-1} = D_n$, so that $D_n = 0$ when $a_n = 0$, we see that when $P(z)$ and $P(-z)$ vanish for $z = 0$ then $D_n = 0$. Supposing that $a_n \neq 0$, we shall show that $P(z)$ and $P(-z)$ have a zero in common if and only if $D_{n-1} = 0$, and the theorem will be proved. Now, $P(z)$ and $P(-z)$ have a zero in common if and only if $P(z) + P(-z)$ and $P(z) - P(-z)$ have a zero in common, *i.e.*, if and only if the polynomials

$$G(z) = a_1 z^{n-1} + a_3 z^{n-3} + \cdots$$

and

$$H(z) = z^n + a_2 z^{n-2} + \cdots$$

have a zero in common. If n is even, $n = 2k$, then

$$G(z) = z(a_1 z^{2k-2} + a_3 z^{2k-4} + \cdots), \quad H(z) = z^{2k} + a_2 z^{2k-2} + \cdots,$$

and if n is odd, $n = 2k+1$, then

$$G(z) = a_1 z^{2k} + a_3 z^{2k-2} + \cdots, \quad H(z) = z(z^{2k} + a_2 z^{2k-2} + \cdots).$$

Since $G(0) \neq 0$ in the second case, and $H(0) \neq 0$ in the first, we conclude that

$G(z)$ and $H(z)$ have a zero in common, when $n=2k$, if and only if the polynomials $a_1w^{k-1}+a_3w^{k-2}+\cdots+a_{2k-1}$ and $w^k+a_2w^{k-1}+\cdots+a_{2k}$ have a zero in common, and, when $n=2k+1$, if and only if the polynomials $a_1w^k+a_3w^{k-1}+\cdots+a_{2k+1}$ and $w^k+a_2w^{k-1}+\cdots+a_{2k}$ have a zero in common. In either case the resultant of the two equations is D_{n-1}, and the theorem is therefore true.

4. Bounds for the moduli of the zeros of $P(z)$. Let us suppose that (1.3) holds with $c_p\neq 0$, $p=1, 2, \cdots, n$, the coefficients of $P(z)$ being complex numbers. We may write (1.3) in the form

$$\frac{Q(z)}{P(z)}=\cfrac{(c_1z+1)^{-1}}{1+\cfrac{h_1}{1+\cfrac{h_2}{1+\cfrac{\ddots}{\ddots+\cfrac{h_{n-1}}{1}}}}},$$

where

$$h_1=1/[c_2z(c_1z+1)],\ h_2=1/(c_2c_3z^2),\ h_3=1/(c_3c_4z^2),\ \cdots,\ h_{n-1}=1/(c_{n-1}c_nz^2).$$

Let $g_1, g_2, \cdots, g_{n-1}$ be any numbers such that $0<g_p<1$, $p=1, 2, \cdots, n-1$. Then if z satisfies the inequalities

$$|h_1|\leqq g_1,\ |h_2|\leqq(1-g_1)g_2,\ |h_3|\leqq(1-g_2)g_3,\ \cdots,\ |h_{n-1}|\leqq(1-g_{n-2})g_{n-1}, \tag{4.1}$$

we are assured by a known theorem* that $P(z)\neq 0$.

For the polynomial $z^2+a_1z+a_2$, (4.1) gives the bound

$$|z|\geqq\frac{|a_1|+(|a_1|^2+[4|a_2|/g_1])^{1/2}}{2}.$$

Inasmuch as g_1 may be taken as near unity as desired, this bound cannot be improved upon.

On putting $g_p=1/2$, (4.1) gives the bound $|z|\geqq c$, where c is the largest of the numbers

$$\frac{(|c_2/c_1|)^{1/2}+(8+|c_2/c_1|)^{1/2}}{2|c_1c_2|^{1/2}},\ 2/|c_2c_3|^{1/2},\ \cdots,\ 2/|c_{n-1}c_n|^{1/2}. \tag{4.2}$$

In particular, if c is the smallest of the numbers $|c_p|$, $p=1, 2, \cdots, n$, then $P(z)\neq 0$ for $|z|\geqq 2/c$.

5. Determination of the number of zeros of $P(z)$ in each of the half-planes $R(z)<0$ and $R(z)>0$. We suppose now that $P(z)$ is a polynomial with real coefficients of the form (1.1), such that the determinants (1.4) are different from zero.

* See [6] pp. 369–372.

Then, by Theorem B, we have the expansion (1.3) where the c_p are different from zero.

THEOREM D. *In the expansion* (1.3), *let* k *of the coefficients* c_p *be negative and the remaining* $n-k$ *be positive. Then* k *of the zeros of* $P(z)$ *have positive real parts and* $n-k$ *have negative real parts.*

Proof. We start with the expansion (3.10), where $G(z)$ and $H(z)$ are the polynomials in (3.9). It is easy to see that

$$G(z) = a_1z^{n-1} + a_3z^{n-3} + a_5z^{n-5} + \cdots,$$

$$H(z) = z^n + a_2z^{n-2} + a_4z^{n-4} + \cdots.$$

If we replace z by $-iz$ and make the substitution (2.1) in (3.10), we obtain after some simple transformations:

$$\frac{G(-iz)}{iH(-iz)} = \cfrac{1}{c_1z - \cfrac{1}{c_2z - \cfrac{1}{c_3z - \cfrac{\ddots}{\ddots - \cfrac{1}{c_nz}}}}}. \tag{5.1}$$

Under the hypothesis that the expansion (1.3) exists, the fraction $Q(z)/P(z)$ is irreducible, and consequently $P(z)$ and $P(-z)$ have no zeros in common. Thus, the zeros of $P(z)$ have their real parts different from zero. For $R(z)=0$ we may therefore write $P(z)=re^{i\theta}$, where $r>0$. If we regard $P(z)$ as the product of the vectors from its zeros to the point z, we then conclude immediately that as z ranges along the axis of imaginaries from $i\cdot\infty$ to $-i\cdot\infty$, θ decreases by the integral amount

$$N - P = \Delta, \tag{5.2}$$

where P and N are the numbers of zeros of $P(z)$ having positive and negative real parts, respectively. The same conclusion evidently results if we introduce an arbitrary constant factor $k\neq 0$, and consider $kP(z)=re^{i\theta}$.

Now,

$$\begin{aligned} i^nP(-iz) &= (z^n - a_2z^{n-2} + a_4z^{n-4} - \cdots) + i(a_1z^{n-1} - a_3z^{n-3} + a_5z^{n-5} - \cdots) \\ &= U(z) + iV(z), \end{aligned}$$

where $U(z)=z^n-a_2z^{n-2}+\cdots$ and $V(z)=a_1z^{n-1}-a_3z^{n-3}+\cdots$ are real when z is real. From the preceding we then conclude that as z increases over the real axis from $-\infty$ to $+\infty$,

$$\theta = (1/\pi) \arctan [V(z)/U(z)]$$

decreases by the amount Δ. Moreover, from (5.1) it follows that

$$\frac{V(z)}{U(z)} = \cfrac{1}{c_1 z - \cfrac{1}{c_2 z - \cfrac{\ddots}{\ddots - \cfrac{1}{c_n z}}}}. \tag{5.3}$$

Let $x_1, x_2, \cdots, x_k$ denote the real distinct zeros of $U(z)$. Let $s_p = +1, 0$, or -1 according as $V(z)/U(z)$ increases from $-\infty$ to $+\infty$ does not change sign, or decreases from $+\infty$ to $-\infty$, respectively, as z increases through the value x_p. We must then have*

$$\Delta = \sum_{p=1}^{k} s_p. \tag{5.4}$$

To compute the number Δ, let the polynomials $f_0 = 1, f_1 = c_n z, \cdots, f_n$ be defined by the recurrence formula

$$f_{p+1} = c_{n-p} z f_p - f_{p-1}, \qquad p = 1, 2, \cdots, n-1, \tag{5.5}$$

and define polynomials $F_0 = 0, F_1 = 1, \cdots, F_n$ by the formulas

$$F_{p+1} = c_{n-p} z F_p - F_{p-1}, \qquad p = 1, 2, \cdots, n-1. \tag{5.6}$$

On multiplying (5.6) by f_p and (5.5) by $-F_p$, and adding, we get $F_{p+1}f_p - F_p f_{p+1} = F_p f_{p-1} - F_{p-1}f_p$, from which we conclude that

$$F_{p+1}f_p - F_p f_{p+1} = 1, \qquad p = 0, 1, \cdots, n-1. \tag{5.7}$$

Consider now the sequence

$$f_0, f_1, f_2, \cdots, f_n. \tag{5.8}$$

From (5.7) it follows that two successive members of this sequence cannot vanish simultaneously. From (5.5) it follows that when f_p vanishes, then f_{p-1} and f_{p+1} have opposite signs ($1 \leqq p \leqq n-1$). Hence, as z increases through a real zero of f_p there can be no loss or gain in the number of variations in signs in the sequence (5.8). Therefore, as z increases through real values from $-\infty$ to $+\infty$, any change in the number of variations must be due to the vanishing of f_n. Moreover, there will be a loss or a gain in the number of variations according as the product $f_{n-1}f_n$ changes from negative to positive or from positive to negative, respectively, as z passes through a zero of f_n. But $f_{n-1}/f_n \equiv V(z)/U(z)$, and consequently the number Δ is precisely the net loss in the number of variations in signs in the sequence (5.8) as z ranges from $-\infty$ to $+\infty$ through real values.

Now when z is negative, the signs of the leading terms in (5.8) are the same as the signs of

* This is the *Cauchy index* of the fraction (5.3). *Cf.* §6.

$$(5.9) \qquad 1, -c_n, +c_{n-1}c_n, -c_{n-2}c_{n-1}c_n, \cdots, (-1)^n c_1 c_2 \cdots c_n,$$

while for positive z, the signs are those of

$$(5.10) \qquad 1, +c_n, +c_{n-1}c_n, +c_{n-2}c_{n-1}c_n, \cdots, +c_1 c_2 \cdots c_n.$$

If there are k variations in signs in (5.10), then there are $n-k$ variations in signs in (5.9). Therefore, $\Delta=(n-k)-k=n-2k$. By (5.2) and the relation $P+N=n$, it then follows that $P=k$ and $N=n-k$. Since k is clearly equal to the number of negative terms in the sequence $c_1, c_2, \cdots, c_n$, Theorem D is established.

Theorem D cannot be applied to a polynomial $P(z)$ in case it has a zero in common with $P(-z)$. This is not a serious difficulty inasmuch as the common factor may be determined by means of the euclidean algorithm. However, there are other cases where the theorem will not apply. For example, if

$$P(z) = z^4 - (9/4)z^3 + z^2 - (9/4)z + (5/2),$$

the determinant $D_2=0$, so that the expansion (1.3) does not exist. The zeros of this polynomial are

$$1,\ 2,\ [-3+i\sqrt{71}]/8,\ [-3-i\sqrt{71}]/8.$$

This difficulty may be avoided in certain cases by considering instead of $P(z)$ the polynomial $(z+s)P(z)$ for a suitable value of s. In the above example, if we take $s=+7/4$, we find that (1.3) exists with $c_1<0$, $c_2>0$, $c_3>0$, $c_4>0$, $c_5<0$.

We remark that if Theorem D is applied to $P(z+h)$ for suitable real values of h, the zeros of $P(z)$ may be separated into sets lying in vertical strips.

Example. Let $P(z)=z^5-3z^4-9z^3-27z^2-32z-30$. Then $Q(z)=-3z^4-27z^2-30$. The expansion (1.3) may be obtained by dividing $P(z)$ by $Q(z)$ until a remainder is obtained which is of lower degree than $Q(z)$; then $Q(z)$ is divided by this remainder, and so on. If we write only the coefficients, the computation may be arranged as follows.

$$\begin{array}{r|l}
 & -1/3+1 \\
-3+0-27+0-30 & 1-3-\ 9-27-32-30 \\
 & \underline{1+0+\ 9+\ 0+10} \\
 & -3-18-27-42-30 \\
 & \underline{-3+\ 0-27+\ 0-30} \quad 1/6 \\
 & -18+\ 0-42+\ 0 \quad -3+0-27+0-30 \\
 & \qquad\qquad -3+0-\ 7+0 \quad 9/10 \\
 & \qquad\qquad \underline{-20+0-30} \quad -18+0-42+0 \\
 & \qquad\qquad\qquad -18+0-27 \quad 4/3 \\
 & \qquad\qquad\qquad \underline{-15+0} \quad -20+0-30 \\
 & \qquad\qquad\qquad\qquad -20+0 \quad 1/2 \\
 & \qquad\qquad\qquad\qquad \underline{-30} \quad -15 \\
 & \qquad\qquad\qquad\qquad\qquad -15 \\
 & \qquad\qquad\qquad\qquad\qquad 0
\end{array}$$

Hence, $c_1=-1/3$, $c_2=1/6$, $c_3=9/10$, $c_4=4/3$, $c_5=1/2$. Therefore, there is one zero in the right half-plane and four in the left half-plane. We find that the zeros are $-1+i$, $-1-i$, 6.197, and the two zeros of the quadratic $z^2+1.92z+2.416$. The smallest modulus of the c_p is $c=1/6$, and therefore an upper bound for the moduli of the zeros is 12. A better bound (*cf.* (4.2)) is the largest of the numbers $3[1+17^{1/2}]/2$, $4(15)^{1/2}/3$, $30^{1/2}/3$, $6^{1/2}$, namely, the first, which is 7.68+. By suitably adjusting the values of the parameters g_p in (4.1), one can obtain a bound very close to the least bound. For example, with $g_1=4/5$, $g_2=4/5$, $g_3=1/2$, $g_4=1/12$, we find that the largest of the numbers corresponding to (4.2) is 6.474.

6. Historical survey.* The problem of determining the number N of zeros of a polynomial $P(z)$ which lie in a given region was solved by Cauchy by means of his formula

$$N=\frac{1}{2\pi i}\int_C \frac{P'(z)}{P(z)}\,dz.$$

Here C is the rectifiable boundary curve of the region, and it must be supposed that $P(z)\neq 0$ on C. If we write $P(z)=re^{2i\pi\theta}$, $r>0$ on C, then it suffices to consider only the real part of the integral, namely

$$N=\int_C d\theta.$$

Hence, N is the amount that θ increases as the point z traverses the curve C in the positive sense. If $P(z)=U(z)+iV(z)$, where $U(z)$ and $V(z)$ are real on C, then

$$\theta=\frac{1}{2\pi}\operatorname{arc\,tan}\,[V(z)/U(z)].$$

These considerations led Cauchy [1] to introduce the notion of the *index* of a rational fraction. The number Δ of (5.4) is the index of the fraction (5.3) for the real axis. Cauchy developed formulas for the computation of the index. He also introduced the method, which we have used in §5, to compute the index by means of a Sturm's series. Kronecker [5] extended the notion of index to systems of functions.

On the basis of Cauchy's work, E. J. Routh [8] derived the following rule for testing a polynomial $P(z)=a_0z^n+z_1z^{n-1}+\cdots+a_n$, $a_0>0$, with real coefficients. Consider the array

$$\begin{array}{llll} a_0, & a_2, & a_4, & \cdots \\ a_1, & a_3, & a_5, & \cdots \\ \dfrac{a_1a_2-a_0a_3}{a_1}, & \dfrac{a_1a_4-a_0a_5}{a_1}, & & \cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots & & & \end{array}$$

* *Cf.* [9].

where the third row is obtained from the first two by cross-multiplication. The next row is obtained from the second and third by the same process. Thus, the first element in the fourth row is

$$\frac{a_3\left(\frac{a_1a_2 - a_0a_3}{a_1}\right) - (a_1a_4 - a_0a_5)}{\frac{a_1a_2 - a_0a_3}{a_1}}.$$

Each row has one fewer elements than the preceding row. Then the number of variations in signs in the sequence making up the first column of the array is equal to the number of zeros of $P(z)$ having positive real parts.

This is essentially Theorem D of §5. In fact, if one starts with the equations (3.4), he will readily see that the test sequence of Routh is, if $a_0=1$:

$$1,\ b_0,\ b_1,\ b_0b_2,\ b_1b_3,\ b_0b_4,\ \cdots, \tag{6.1}$$

which is equivalent to $1,\ c_1,\ c_1c_2\cdots,\ c_1c_2\cdots c_n$. The number of variations in signs in this sequence is equal to the number of negative terms in the sequence $c_1,\ c_2,\ \cdots,\ c_n$. The method of Routh fails in case division by zero is involved in his algorithm. Theorem A shows that the method cannot fail in the important case where the zeros all have negative real parts. In terms of the determinants D_p one sees by (3.7) and (6.1) that Routh's test sequence is $1, D_1, D_2/D_1, \cdots, D_n/D_{n-1}$.

Hermite [3] expressed Cauchy's index in terms of the signature of a certain quadratic form, and obtained necessary and sufficient conditions for the zeros of a polynomial with complex coefficients to lie in a half-plane. For the function (5.3), this quadratic form can be written, following A. Hurwitz [4], as

$$F_m(X, X) = \frac{1}{2\pi i}\int \frac{V(z)}{U(z)}\left[X_0 + X_1z + \cdots + X_{m-1}z^{m-1}\right]^2dz,$$

where the integral is taken around a curve containing all the zeros of $U(z)$. If we write $V(z)/U(z)=\sum_{p=0}^{\infty}k_p/z^{p+1}$, then an easy computation shows that

$$F_m(X, X) = \sum_{p,q=0}^{m-1} k_{p+q}X_pX_q.$$

Hence, if

$$\Delta_p = \begin{vmatrix} k_0, & k_1, \cdots, & k_{p-1} \\ k_1, & k_2, \cdots, & k_p \\ \cdots & \cdots & \cdots \\ k_{p-1}, & k_p, \cdots, & k_{2p-2} \end{vmatrix} \neq 0, \qquad p = 1, 2, \cdots, n,$$

then

$$F_n(X, X) = \Delta_1Y_0^2 + [\Delta_2/\Delta_1]Y_1^2 + \cdots + [\Delta_n/\Delta_{n-1}]Y_{n-1}^2,$$

where the Y_p are linear functions of the X_p. The Cauchy index Δ is the signature

of this quadratic form. Now, the coefficients c_p in the continued fraction (5.3) can be expressed in terms of the Δ_p by well-known formulas.* In this way one may obtain Theorem D without using the Sturm's series (5.8). This is essentially the method used by Hurwitz to prove that all the zeros of $P(z)$ have negative real parts if and only if the determinants (1.4) are all positive. This theorem is an immediate consequence of Theorems A and B.

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* *Cf.*, for instance, [11].

Mathematicians and philosophers. In my opinion a mathematician, in so far as he is a mathematician, need not preoccupy himself with philosophy—an opinion, moreover, which has been expressed by many philosophers.—Henri Lebesgue (1936).

The judgment of oratory. In the pure mathematics we contemplate absolute truths which existed in the divine mind before the morning stars sang together, and which will continue to exist there when the last of their radiant host shall have fallen from heaven.—Edward Everett (1870).

The judgment of reason. Everything else they have tried to fasten onto mathematics—its absoluteness and its complete exactitude, its generality and its independence, in a word its truth and everlastingness—all this (I may be pardoned the strong language), *is a pure superstition!*—Gerritt Mannoury (1934). *Contributed.*

NUMBERS GENERATED BY THE FUNCTION e^{e^x-1}

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1. Introduction. The integers which are the subject of this paper have been discussed cursorily by several writers* who contented themselves with discovering Lemma 1 and Corollary 1, and computing the first few. They occur in combinatory analysis, being, in fact, the sum of the horizontal entries in the table of p. 169 of Netto's *Lehrbuch der Combinatorik*. Their interest is primarily number-theoretic. Indeed, from Minetola's† work, it is evident that G_n (as defined below) is the number of ways in which a product of n distinct primes can be factored. Thus, $p_1p_2p_3=(p_1p_2)p_3=(p_1p_3)p_2=(p_2p_3)p_1=(p_1)(p_2)(p_3)$, and so $G_3=5$.

It will be convenient to give an algebraic definition of G_n.

2. Definition and algebraic properties. We determine the sequence of G's (we shall only be interested in the case where n is a non-negative integer) by the following

DEFINITION.

$$G_n e = \sum_{r=0}^{\infty} \frac{r^n}{r!} \qquad (n = 0, 1, \cdots).$$

It is plain from this that

$$G_0 = G_1 = 1.$$

More general summation are expressible in terms of the G's; in fact

THEOREM 1.

$$\sum_{r=0}^{\infty} \frac{(ar+b)^n}{r!} = (aG+b)^n e,$$

where the right-hand member means

$$e\sum_{r=0}^{n}\binom{n}{r}a^r G_r b^{n-r}.\ddagger$$

For, expanding the first member, we find it equals

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*Wohlsentolme gave just this as a problem, on the Tripos: to prove Lem. 1, Cor. 1, and find G_8. See, Bromwich, Infinite Series, p. 197.

† Silvio Minetola, Principii di Analisi Combinatoria, Jior. di Mat., vol. 45, 1907, pp. 333–366, vol. 47, 1909, pp. 173–200.

‡ We shall make considerable use of this standard symbolic convention: whenever we write a "product" of G's, say,

$$G^jG^kG^m\cdots,$$

we mean the sum of the exponents to be taken as a subscript

$$G_{j+k+m+\cdots}.$$

$$\sum_{r=0}^{\infty}\frac{1}{r!}\sum_{s=0}^{n}\binom{n}{s}a^s r^s b^{n-s}=\sum_{s=0}^{n}\binom{n}{s}a^s b^{n-s}\sum_{r=0}^{\infty}\frac{r^s}{r!}.$$

LEMMA 1. $G_{n+1}=(G+1)^n$ $(n=0, 1, \cdots)$.

Write $a=b=1$ in Theorem 1 and multiply numerator and denominator of the summand by $(r+1)$ to reduce it to the form of the definition. This proposition, together with the fact that the first two G's are integers, insures that all the G's are positive integers.

The lemma gives us a ready method of computing the numbers. We find

$G_0 = 1$	$G_5 = 52$	$G_{10} = 115975$
$G_1 = 1$	$G_6 = 203$	$G_{11} = 678570$
$G_2 = 2$	$G_7 = 877$	$G_{12} = 4213597$
$G_3 = 5$	$G_8 = 4140$	$G_{13} = 27644437$
$G_4 = 15$	$G_9 = 21147$	$G_{14} = 190899322$

and so on.

It is evident that the G's increase very rapidly; this is reflected in

THEOREM 2. $G_n \geqq k^n/k!$ $(k=0, 1, \cdots)$.

For $k=0$ this is trivial; if it is true for some k, we have

$$G_n=\sum\binom{n-1}{r}G_r \geqq \sum\binom{n-1}{r}\frac{k^r}{k!}=\frac{(k+1)^n}{(k+1)!}.$$

Unfortunately G_{n+1} is not a very good upper bound for the function $x^n/\Gamma(x)$; *e.g.*, the greatest value attained by the function $x^3/\Gamma(x)$ is approximately 13.56.

Lemma 1 generalizes inductively to

THEOREM 3.

$$\sum_{r=0}^{k}(-)^r\binom{k}{r}G_{n-r+1}=\sum_{r=0}^{n-k}\binom{n-k}{r}G_{n-r} \qquad (0 \leqq k \leqq n).$$

This is certainly valid for $k=0$. Now, let $F(n, k)$ denote either member of the equation which we assume true for this particular k. Then, by the well known properties of the binomial coefficients, we have

$$F(n, k) - F(n-1, k) = F(n, k+1),$$

which establishes the theorem, by induction on k.

The case $k=n$ is of some interest.

COROLLARY 1. $G_n=G(G-1)^n$.

We justify the title of the paper by

THEOREM 4. $e^{e^x-1}=e^{Gx}$,

where, again, e^{Gx} is the symbolic representation of

$$\sum_{r=0}^{\infty} G_r x^r/r!.$$

For,

$$e^{e^x}=\sum_r \frac{1}{r!}\sum_s \frac{r^s x^s}{s!}=\sum_s \frac{x^s}{s!}\sum_r \frac{r^s}{r!}.$$

This theorem gives rise to a recurrence relation for the G's which is quite different from any we have yet stated.

THEOREM 5. $G(G-1)\cdots(G-n+1)=1 \qquad (n=0, 1, \cdots).$

We shall adopt the notation S_j^i to stand for the sum of all possible products of the numbers $1, 2, \cdots, (i-1)$, taken j at a time. Then, by a change of variable ($x=\log(1+u)$) in Theorem 4,

$$e^x=\sum \frac{G_r}{r!}\log^r(1+x)=\sum_r G_r \sum_n (-)^{n-r} S_{n-r}^n \frac{x^n}{n!}$$

$$=\sum_n \frac{x^n}{n!}\sum_r (-)^{n-r} S_{n-r}^n G_r,$$

whence,

$$\sum_{r=0}^{n} (-)^{n-r} S_{n-r}^n G_r = 1.$$

There is also an elegant symbolic proof of this.

$$e^x = e^{G\log(1+x)} = (1+x)^G = \sum_{n=0}^{\infty} G\cdots(G-n+1)\frac{x^n}{n!}$$

3. Number-theoretic properties. We are now in a position to discuss the curious arithmetic properties which these numbers possess.

LEMMA 2. $G_p \equiv 2 \pmod p$,

where, as in all the following theorems, p is a prime number.

Write $n=p$ in Theorem 5. By Lagrange's proof of Fermat's Theorem, all the intermediate coefficients are congruent to zero, modulo p; the first to $+1$; and the last to -1.

This proposition is the starting point for a series of inductions, which culminates in Theorem 6.

LEMMA 3. $G_{p+n} \equiv G_n + G_{n+1} \pmod p$.

When $n=0$ this reduces to Lemma 2. Write $p+n-1$ for n, and p for k, in Theorem 3; then

$$G_{p+n} - G_n \equiv \sum \binom{n-1}{r} G_{p+n-r-1} \pmod{p}.$$

Now, assume that, for all integers $<n$ the statement is true. We then have

$$G_{p+n} - G_n \equiv \sum \binom{n-1}{r} G_{n-r-1} + \sum \binom{n-1}{r} G_{n-r} \pmod{p}$$

$$= \sum \binom{n}{r} G_{n-r} = G_{n+1}.$$

Letting $p=2$ we obtain the significant

COROLLARY 2. $G_n+G_{n+1}+G_{n+2}\equiv 0 \pmod 2$.

LEMMA 4. $G_{kp^s+n}\equiv G^n(G+s)^k \pmod p$.

We shall show that its truth for fixed s, all n, and $k=1$, implies its truth for the same s, all n, and all k. For, assume it for some s and k, and all n; then

$$G_{(k+1)p^s+n} = G_{kp^s+(p^s+n)} \equiv G^{p^s+n}(G+s)^k \pmod p$$
$$\equiv G^n(G+s)(G+s)^k = G^n(G+s)^{k+1}.$$

Now, assume the theorem for all k, all n, and some s. Writing $k=p$,

$$G_{p^{s+1}+n} \equiv G_{p+n} + s^p G_n \equiv G_{n+1} + (s+1)G_n \pmod p,$$

which, by our preliminary remark, implies its validity for all k, all n, and $s+1$. Since the statement is obviously correct for $s=0$, it is universally true by induction on s.

If the subscript of G is not of the form of the lemma, but is a polynomial in p, it can be reduced by considering everything after the leading term as n. Repeated application of the proposition yields

THEOREM 6. $G_{\sum k_r p^r}\equiv \Pi(G+r)^{k_r} \pmod p$,

where the limits of the summation and product are the same.

THEOREM 7. *The G's have a "congruence-period" of* $(p^p-1)/(p-1)$ *places; i.e.,*

$$G_{n+(p^p-1)/(p-1)} \equiv G_n \pmod p \qquad (n = 0, 1, \cdots).$$

For, by Theorem 6, we have

$$G_{p^{p-1}+\cdots+p+1+n} \equiv G^{n+1}(G+1)\cdots(G+p-1) \pmod p$$
$$\equiv G_{p+n} - G_{n+1} \equiv G_n.$$

We have shown therefore that the smallest period of the least residues of the G's is a divisor of $(p^p-1)/(p-1)$. Indeed, when $p=2$, 3, or 5, it is precisely this, although for $p=5$, either of the factors of 781, 11 and 71 might be a priori candidates. The author has succeeded in computing the least residue pattern in these

three cases, but since the latter is rather excessive, we give only the first two. For 2, it is

$$1,\ 1,\ 0$$

and for 3

$$1,\ 1,\ 2,\ 2,\ 0,\ 1,\ 2,\ 1,\ 0,\ 0,\ 1,\ 0,\ 1.$$

For primes >5 the situation is unknown.

We close with what is, in view of Theorem 7, a natural generalization of Corollary 2. We are indebted to Dr. Irving Kaplansky for the proof.

THEOREM 8. *The sum of* $(p^p-1)/(p-1)$ *consecutive G's is a multiple of* p.

We pass to the Galois field of p elements, so that congruence modulo p becomes equality. Solving the difference equation

$$G_{n+p} - G_{n+1} - G_n = 0$$

by standard methods, we find

$$G_n = a_1x_1^n + a_2x_2^n + \cdots + a_px_p^n$$

where $x_1, \cdots, x_p$ are the (distinct) roots of

$$x^p - x - 1 = 0.$$

Now, precisely as in the case of the G's, we find

$$x^{p^n} = x + n,$$

so that

$$x^{(p^p-1)/(p-1)} = x(x+1)\cdots(x+p-1) = x^p - x = 1,$$

whence

$$1 + x + \cdots + x^{(p^p-1)/(p-1)-1} = \frac{1-1}{x-1} = 0$$

which proves the theorem.

DESCRIPTIVE GEOMETRY AS USED IN THE SLAUGHTER HOUSE

(As reported in the Kansas City Star to have been quoted from a 24-page booklet from the Office of Price Administration)

"Then all fat shall be removed which extends above a flat plane using the following two lines as guides for each edge of the plane: an imaginary line parallel with the full length of the protruding edge of the lumbar section of the chine bone which line extends one inch directly above such protruding edge; a line on the inside of the loin two inches from the flank edge, and running parallel with such edge for the full length of the loin."

Shades of Taurus—can it be that our classes soon will be filled with embryo butchers and meat-cutters, learning the rudiments of their trade via courses in descriptive geometry?

W. B. Campbell.

CLUBS AND ALLIED ACTIVITIES

EDITED BY J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to J. S. Frame, Michigan State College, East Lansing, Michigan.

SOME SLIDE RULES

L. R. FORD, Illinois Institute of Technology

1. Introduction. The material to follow was presented twice during 1944 to mathematical clubs made up largely of high school teachers.* The rules that are described are, for the most part, applicable to problems in high school mathematics; and rules made by high school students were exhibited and used. Their design and construction serve to stimulate the interest of the student and to make demands upon his ingenuity.

The underlying theory of these devices is very simple. Unfortunately it is ordinarily not readily available in the students' text books. It is because of this situation, and at the request of several of the teachers, that this description is being written.

2. Scales. The theory of the slide rule is based on the concept of the scale. Familiar examples of scales are the common foot rules, thermometers, and the like, with the graduations marked upon them. These exemplify the scale as a line with certain marks upon it to which numbers are attached.

If we mark an origin O on the line and locate points of the line by their distances (with suitable sign) from O, then the distance X to the point marked x depends upon x. That is, X is a function of x,

$$X = f(x). \tag{1}$$

Conversely, we may assume some function $f(x)$ and construct a scale from it. Thus we take $x = 0, 1, 2, \cdots$, or whatever values we are interested in, and compute the corresponding distances X. We lay these distances off from O, mark the points, and attach the values of x.

By taking various functions in (1) we can form the greatest variety of scales. The uniform scales of foot rule and thermometer arise from relations of the form $X = ax + b$. The scales of the logarithmic slide rule are based on the function $X = \log x$. The scales to be used in what follows will depend upon the diverse functions appearing in the problems to be solved.

3. The slide rule. The slide rule is a device for adding distances. Because the distances are those involved in scales the result is the addition of functions.

* Men's Mathematics Club, Chicago and Metropolitan Area, May 19, 1944. Women's Mathematics Club, November 4, 1944.

Consider two material objects, such as strips of wood or of pasteboard, which can be placed on opposite sides of a line along which they can slide freely. Place the objects together in an initial position and mark the common origin O upon the line. Measuring from O we construct upper and lower scales,

$$X = f(x), \qquad X = F(y), \tag{2}$$

respectively.

Now, let one scale slide along the other to a new position, as in Figure 1. Let z in the upper scale coincide with y in the lower scale; and let u in the upper coincide with v in the lower. Since the distance from the mark z to the mark u is equal to the distance from the mark y to the mark v we have the equation

$$f(u) - f(z) = F(v) - F(y). \tag{3}$$

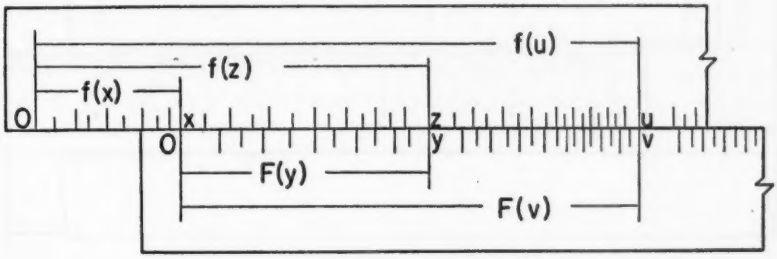

FIG. 1.

This rule enables us to solve equation (3) for any one of the four variables when the other three are given. For example, suppose u, v, z are given and it is required to find y. We bring u on the upper scale into coincidence with v on the lower scale; then opposite z on the upper scale we read the desired value of y on the lower scale. The problem is thus solved by one setting of the rule.

An equation in three variables results if the initial mark O, corresponding to a zero value of one of the functions, is used. Thus clearly, from the figure, we have

$$f(z) = f(x) + F(y). \tag{4}$$

We can solve (4) for any of the three variables when the other two are given by one setting of the rule.

Many useful rules employ the same scale function for both scales. This will be true of the examples to be presently shown. If $f(x)$ is the function used then (3) takes the form

$$f(y) + f(u) = f(z) + f(v); \tag{5}$$

and (4) becomes

$$f(z) = f(x) + f(y). \tag{6}$$

4. The Pythagorean rule. The rule shown in Figure 2, which uses the scale function $X=x^2$, is one of the most instructive for the beginner. It is not difficult to construct and its uses are legion. We lay off from O distances, 1, 4, 9, 16, $\cdots$, 100, and mark them 1, 2, 3, $\cdots$, 10. We mark the fractional points of division but do not number them. Upper and lower scales are alike.

The figure shows how a folded piece of cardboard and a long strip sliding in the groove make up a fairly good mechanism. Each part carries a scale and the two scales slide freely along one another.

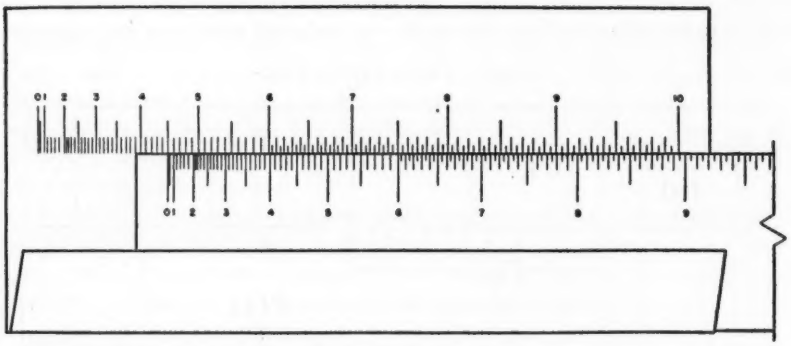

FIG. 2. $x^2+y^2=z^2$

Experience will suggest convenient ways of making the scales. Since the scales are alike they can be made by drawing markings across a line after which they are cut apart along this line and pasted in their places. Rough scales may be very rapidly made by drawing them on the white back of ruled paper held against a window. The rulings are visible while the scale is being made but are not visible when pasted on the rule. Good rules, however, take more care than this. In any case, a table of squares is very useful in the construction of this rule.

Equations (5) and (6) now become

$$y^2 + u^2 = z^2 + v^2, \tag{7}$$

and

$$z^2 = x^2 + y^2. \tag{8}$$

We can use the rule to solve for any one of the variables in either of these equations when the remaining variables are given.

Let us observe, first of all, that the numbers on the rule may all be multiplied by the same factor. Thus, 5, 6, 7, *etc.*, may be taken to represent 50, 60, 70, *etc.*, or, in general, $5k$, $6k$, $7k$, *etc.* This results from the fact that the equation satisfied by the variables is homogeneous in the variables. For the same reason, the numbers on each of the two rules described in the following sections may be

multiplied by a common factor. This change of scale is often resorted to in order to bring desired values within the range of the slide rule.

We give some problems which are solved by the rule as set in the figure.

What is the distance to the top of a tower 45 feet high from a point on the level ground 70 feet from the base?

We set 0 under 45 and read above 70 the answer, 83.2.

A force is the resultant of perpendicular components of 150 pounds and 160 pounds. How may it be broken into perpendicular components one of which is 120 pounds?

Here we divide by 2 to get convenient values. Setting 60 under 75 we read above 80 the value 91.8. The answer is twice this, or 183.6 pounds.

If $\sin\theta = .45$, what is the value of $\cos\theta$?

Since $\sin^2\theta + \cos^2\theta = 1$, we set 0 under .45 and read the answer under 1.00; namely, $\cos\theta = .894$.

Many other interesting applications will occur to the student. For example, knowing the radii of two circles and the distance between their centers he can read off the lengths of the internal and external tangents. He will be gratified by the rapidity with which he can get numerical answers based on applications of the theorem of Pythagoras.

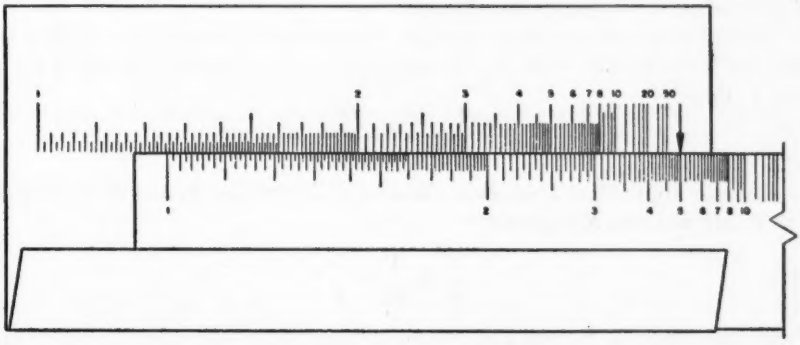

FIG. 3. $1/x+1/y=1/z$.

5. The Share-the-Work rule. If A can do a piece of work in x days and B can do it in y days, how long will it take them to do it working together?

Letting z be the required number of days, we add the fractional part of the whole that is done in a day by A to the part done by B to get the part done by both together:

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{z}. \tag{9}$$

In Figure 3 is shown a rule for solving this problem. It is based on the scale

function, $X = 1/x$, both above and below. The position marked O in earlier scales and from which distances are measured is marked here by an arrow. Distances are considered positive to the left. This was suggested by a high school student with the object of having the natural numbers run in their conventional order from left to right.

If A does the work in 5 days and B does it in 7 days we learn from the rule (reading just below 7) that both together will do it in 2.9 days.

This rule is useful in the study of lenses and mirrors. The distance p of an object from lens or mirror and the distance q of the image are connected with the focal length f by the equation

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{f}. \tag{10}$$

For example, if an object is 5 cm. from a lens of focal length 4 cm., we read from the rule that the image will be at a distance 20 cm. from the lens.

If thin lenses of focal lengths f_1 and f_2 are in contact, the combination is equivalent to a single lens of focal length F, where

$$\frac{1}{f_1} + \frac{1}{f_2} = \frac{1}{F}. \tag{10'}$$

This is adapted to solution by the rule.

A similar formula arises in aviation. The radius of action R of a plane is the distance it can go and return in one hour. If the ground speeds out and in are V_0 and V_1 then

$$\frac{1}{V_0} + \frac{1}{V_1} = \frac{1}{R}. \tag{11}$$

Another application appears in the case of resistances R_1 and R_2 in parallel. The total resistance R is given by

$$\frac{1}{R_1} + \frac{1}{R_2} = \frac{1}{R}. \tag{12}$$

Thus, resistances of 5 ohms and 3 ohms in parallel are equivalent (reading just below 3) to a total resistance of 1.87 ohms.

Finally, the capacitance of condensers connected in series combine according to the same law.

6. The Hypocycloid rule. If a line of length a moves with its ends on the x- and y-axes (Figure 4) the envelope is the hypocycloid of four cusps

$$x^{2/3} + y^{2/3} = a^{2/3}. \tag{13}$$

We shall assume this result from the calculus. It may be verified by finding the length of the tangent to (13) at a point (x_1, y_1),

$$x_1^{-1/3}x + y_1^{-1/3}y = a^{2/3}, \tag{14}$$

between the intercepts on the axes.

A rule for solving (13) is shown in Figure 7, the scale function being $X=x^{2/3}$ on both parts. This rule was described in the author's article on "Alignment Charts," Notre Dame Mathematical Lectures, No. 4, 1944. It solves various problems in the calculus.

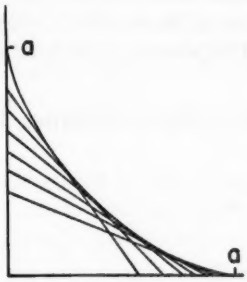

FIG. 4.

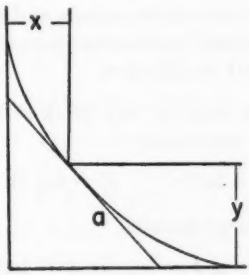

FIG. 5.

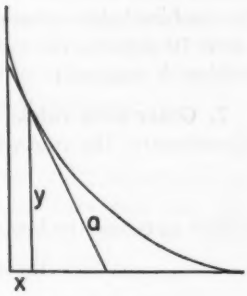

FIG. 6.

Find the length of the longest girder that can be moved from a corridor 6 feet wide into another 8 feet wide at right angles to it.

Figure 5 shows that we are led to the solution of (13). If the envelope of the line passes above the corner the line cannot make the turn; if it passes below there is room to spare. We are concerned therefore with the curve which passes through the corner. Setting 0 under 6 and reading above 8 we find 19.7 feet. The equation of the line in which the girder lies in the extreme position is given by (14).

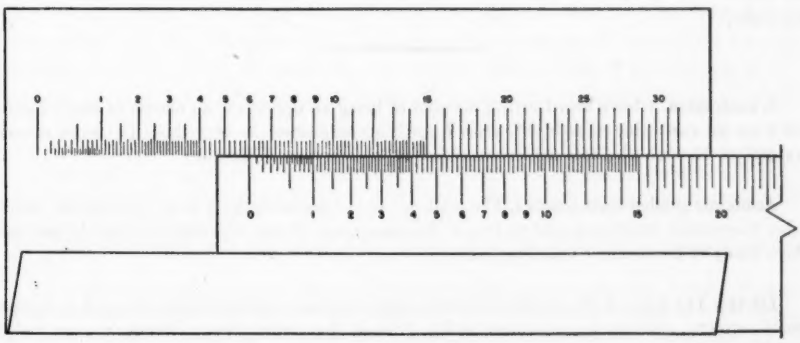

FIG. 7. $x^{2/3}+y^{2/3}=a^{2/3}$.

Find the length of the shortest ladder which will pass over a wall 6 feet high and lean against a wall 1.2 feet back of the first wall. Figure 6 shows that we have here the same sort of problem as before. Above 1.2 on the rule we read the answer 9.3 feet. The position of the ladder is found from (14).

This rule can be used to solve the problem of the shortest straight road across the corner of a rectangular field so as to pass a spring whose location is given.

Surfaces of similar solids are proportional to the two-thirds powers of the volumes. For example, what is the weight of a man whose skin area is equal to the combined skin areas of a boy of 60 pounds and a boy of 70 pounds? Calling a unit 10 pounds, the rule is set to give the answer, 184 pounds. This kind of problem is suggestive of basal metabolism.

7. Other slide rules. The student will think of many other possibilities. In trigonometry, the rule with the scales

$$X = \log x, \qquad X = \log \sin y$$

enables us to use the law of sines rapidly,

$$\log a - \log b = \log \sin A - \log \sin B.$$

A rule with the scales

$$X = \log x, \qquad X = \log \tan y$$

enables us to use the law of tangents; and so on.

Returning to Figure 1, it is clear that the scales where u and v are read may be made with different scale functions, $g(u)$ and $h(v)$. We thus have

$$g(u) - f(z) = h(v) - F(y).$$

The origins for the u- and v-scales should lie together initially, but need not be the same as the origins for the z- and y-scales. Further, they need not lie along the same line as before, but may slide along one another on some other part of the rule.

A confession. I have heard myself accused of being an opponent, an enemy of mathematics, which no one can value more highly than I, for it accomplishes the very thing [a theory of color to supplant Newton's] whose achievement has been denied me.—Goethe.

Academic applied mathematics. The tendency of the mathematician is to overrate the solidity of his theoretical structures, and to forget the narrowness of the experimental foundation upon which many of them rest.—Lord Rayleigh.

Litotes. The mind of the mathematician is subject to many disturbing causes, such as fatigue, loss of memory, and hasty conclusions; and it is found that, from these and other causes, mathematicians make mistakes.—James Clerk Maxwell.

Pythagoras returns. Nature is the numerable.—Spengler.

A geometer on algebra. Algebra which cannot be translated into good English and sound common sense is bad algebra.—W. K. Clifford. *Contributed.*

DISCUSSIONS AND NOTES

EDITED BY MARIE J. WEISS, Sophie Newcomb College, New Orleans 18, La.

The Department of Discussions and Notes is open to all forms of activity in college mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

THE ALTITUDES OF A TRIANGLE AND OF A TETRAHEDRON*

VICTOR THÉBAULT, Tennie, Sarthe, France

THEOREM 1. *In a triangle ABC having altitudes AA', BB', CC', the perpendiculars drawn from the midpoints of $B'C'$, $C'A'$, $A'B'$ to the sides BC, CA, AB respectively, are concurrent at the radical center of the circles (A, AA'), (B, BB'), (C, CC') having centers A, B, C and radii AA', BB', CC' respectively.*

In fact, the circles (B, BB'), (C, CC') pass respectively through B', C', and the circle on BC as diameter passes through both B' and C'. The midpoint of $B'C'$ lies on the radical axis of the two circles, for if the line $B'C'$ meets these circles again in B'' and C'', the midpoint of $B'C'$ coincides with that of $B''C''$, and the theorem is established.

Remarks. The radical axes of the pairs of circles (B, BB') and (C, CC'), (C, CC') and (A, AA'), (A, AA') and (B, BB'), being parallel to AA', BB', CC', bisect three of the interior or exterior angles of the complementary† triangle of $A'B'C'$, so that *the radical center, Ω, of the three circles under consideration coincides with the center of one of the Taylor circles‡ of the fundamental triangle ABC.* Moreover, *the circle (Ω), orthogonal to the circles (A, AA'), (B, BB'), (C, CC'), belongs to the coaxal system determined by the circumcircle and the Lemoine line of triangle ABC.* Finally, Ω coincides with the radical center of the circles having centers A, B, C (or B, C, H) and tangent to the three sides of triangle $A'B'C'$, according as angle A is acute or obtuse, H being the orthocenter of triangle ABC.§

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* Translated from the French by Howard Eves. The footnotes are due to the translator.

† Commonly called the *medial* triangle of $A'B'C'$; *i.e.*, the triangle formed by connecting the midpoints of the sides of $A'B'C'$.

‡ In fact, Ω coincides with the center of the circle well-known as *the* Taylor circle of the triangle ABC.

§ This last statement is in error and should read: *Finally, Ω coincides with the radical center of the three circles having centers A, B, C and tangent to the three sides of triangle $A'B'C'$.* When A, B, C are all acute these circles escribe $A'B'C'$; when A, B, or C is obtuse, two of these circles escribe $A'B'C'$ and one (that corresponding to the obtuse angle) inscribes $A'B'C'$. The Taylor circle of triangle ABC is, in either case, the radical circle of these three circles.

The literature dealing with Taylor's circle contains many confusing statements with regard to the acuteness and obtuseness of the given triangle. Thus, theorems 109 and 115 of Chapter I in Coolidge's A Treatise on the Circle and the Sphere (1916) are not true, as stated, if the fundamental triangle is obtuse.

THEOREM 2. *In a tetrahedron $ABCD$ having altitudes AA', BB', CC', DD', the planes perpendicular to the edges BC, CA, AB, DA, DB, DC and passing through the midpoints of $B'C'$, $C'A'$, $A'B'$, $D'A'$, $D'B'$, $D'C'$ respectively, are concurrent at the radical center of the spheres (A, AA'), (B, BB'), (C, CC'), (D, DD') having centers at A, B, C, D and radii AA', BB', CC', DD' respectively.*

In fact, the spheres (B, BB'), (C, CC') pass respectively through B', C', and the sphere on BC as diameter passes through both B' and C'. The midpoint of $B'C'$, then, lies on the radical plane of the two spheres, for if the line $B'C'$ cuts these spheres again in B'', C'', the midpoints of $B'C'$ and $B''C''$ coincide, and the theorem is proved.

Remarks. The sphere (Ω), orthogonal to the four spheres (A, AA'), (B, BB'), (C, CC'), (D, DD'), belongs to the coaxal system determined by the circumsphere of the tetrahedron $ABCD$ and the harmonic plane of the point K for which the sum of the squares of the distances from the planes of the faces is a minimum (*the first Lemoine point*). When the tetrahedron $ABCD$ is equifacial,* the point Ω coincides with the radical center of the spheres inscribed in the truncs of the tetrahedron.

THE TRANSCENDENTAL CHARACTER OF COS X

R. W. HAMMING, University of Louisville

This note contains an entirely elementary proof of a theorem that does not seem to be well known.†

THEOREM. *The value of* $\cos(a^\circ\, b'\, c'')$, *where a, b, and c are integers, is an algebraic number.*

The proof is based on De Moivre's theorem:

$$(\cos x + i \sin x)^k = \cos kx + i \sin kx.$$

Taking real parts and noting that only even powers of sines occur we have

$$\cos kx = P_k(\cos x),$$

where $P_k(\cos x)$ is a polynomial in $\cos x$ with integral coefficients. Put $k = 90 \cdot 60^2$ and $x = 1''$. Then

$$\cos 90^\circ = 0 = P_k(\cos 1'').$$

Thus $\cos 1''$ is an algebraic number. Now using $k = 60^2 a + 60b + c$ we have

$$\cos(a^\circ\, b'\, c'') = P_k(\cos 1'')$$

and hence it is also algebraic.

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* Such a tetrahedron is commonly referred to as an *isosceles* tetrahedron. See N. A. Court's Modern Pure Solid Geometry, art. 307, page 98.

† D. H. Lehmer, A note on trigonometric algebraic numbers, this MONTHLY, vol. 40, 1933, pp. 165–166.

The fact that the values of the trigonometric functions are algebraic numbers for all rational angles *in degrees* (the method of proof actually shows this) does not contradict the fact that the trigonometric functions are transcendental functions. The one statement says that at certain points the values are algebraic (of varying degrees), while the other says that no single algebraic equation $f(x, y) = 0$ is satisfied by a trigonometric function for all x.

RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.

Nautical Mathematics and Marine Navigation. By S. A. Walling, J. C. Hill, and C. J. Rees. Cambridge, The University Press; New York, The Macmillan Company, 1944. 9+221 pages. $2.00.

This book is planned primarily for those who are anticipating some kind of sea service and find it advisable to review elementary mathematics and practice its application to problems of seamanship.

Part I, indicated by the title, includes a set of interesting and practical problems solvable by the four fundamental operations of arithmetic and simple geometric constructions. These problems range over a remarkably large number of subjects and their solution would make excellent preparation for nearly any work that would be encountered in port or at sea in any kind of ship.

Part II extends the use of the same elementary principles to the simpler problems of piloting but does not go into the field of celestial navigation.

It takes up such subjects as the history and description of the ships compass, the sextant and its use in measuring horizontal angles, variation, deviation, Mercator projection, lines of position, fixes, running fixes, and the use of vectors in determining the effects of currents. All these discussions are supplemented by well selected typical problems. The reading material is illustrated by sixty or more diagrams or drawings. The appendix, consisting of a table of square roots for use in visibility problems and a practice chart centering at Falmouth, England, is followed by a list of answers to all problems.

The Anglo-American authorship introduces a few terms and expressions which may not be familiar to all readers, but for those interested in ships in any way the study of this book will mean time spent in an entertaining and profitable way.

F. E. CARR

Aircraft Analytic Geometry. By J. J. Apalategui and L. J. Adams. New York, McGraw-Hill Book Company, Inc., 1944. 18+285 pages, $3.00.

As stated in the preface, this "book constitutes a new approach to a certain class of problems which arise in the engineering, lofting, and tooling of airplanes." The new approach consists in the application of plane and solid analytic geometry to certain aircraft problems, some of which are enumerated below. It is further stated that "the book is intended for use by men in the lofting, tool designing, and jig building departments and will also be very useful for men in the layout and development groups of the engineering department." Students of descriptive geometry will be interested in the methods given to simplify the checking of layouts made by descriptive geometry. We shall give a brief résumé of each chapter.

Chapter 1 treats of the elements of plane trigonometry, including fundamental identities, interpolation, functions of angles which differ by a multiple of 90°, a chart of 32 figures for solving right and oblique triangles, *etc.* The graphs of $\sin x$ and $\cos x$ on page 12 are rather inaccurate as measurement of $\sin 45°$ and the slope at the origin show.

Chapter 2 consists of the plane analytic geometry of the straight line. Some formulas and relations are derived while others are stated without proof. A few so-called derivations are meagre. For example, on page 27 the point-slope equation is derived by merely substituting x and y for x_2 and y_2 in the formula for the slope. Just a few words of explanation would clarify this, or else the "derivation" might well be omitted entirely.

Chapter 3, Cartesian Coördinates, is an introduction to three-dimensional rectangular coördinate systems as applied to the airplane and its parts. First a system of rigged axes (axes in flying position) is set up for the airplane as a whole. Two principal wing designs are described, and a system of coördinate axes is defined for each. Then, for convenience in analyzing, the airplane is divided into the fuselage, the wings, the nacelles, and the empennage. Axes are defined for each of these divisions, and their relations to the rigged axes are discussed.

Chapter 4, True Lengths and True Angles, comprises an introduction to the space geometry of the straight line, the direction ratios forming the basic concept. (Incidentally, it would seem to be a simplification in conception to speak merely of the direction numbers 2, 4, 5 of a line rather than of the direction ratios 2:4:5, since we need never be concerned with the ratios 2:4 or 2:5; most modern texts in analytic geometry appear to agree with this suggestion.) The formula for "true length" (distance) is derived, direction cosines introduced, and theorems on projections discussed and then applied to the angle between two lines.

Chapter 5 applies the principles of Chapter 4 to determine the direction numbers of various wing axes with respect to the rigged axes, as well as to find the true angle between a line and a plane or between two planes under various hypotheses.

Chapter 6, Equations of Planes, starts with the normal form and then verifies it by a projection theorem of Chapter 4. The usual forms of plane-equations are

given with more than usual emphasis on the determination of angles, this apparently being of great importance in aircraft construction.

Chapter 7, Equations of Lines, gives the standard forms and shows how to compute the distance from a line to a point and between two skew lines.

Chapter 8, Translation and Rotation of Axes, gives derivations of the rotation formulas for transforming from various sub-assembly axes to the rigged axes. The tabular form given in each case has the advantage of being immediately adaptable for transformation from either system of axes into the other.

Chapter 9 deals with various applications of the principles of the preceding chapters. Some of these include the checking of the solution of a problem solved by descriptive geometry and the determination of the angles between certain basic planes in the wing or in the fuselage. The preparation of basic data tables, the master diagram, and the loft layout are explained and illustrated, and various uses of these are given.

Chapter 10 gives a brief treatment of the projective geometry of conic sections. The Theorems of Pascal and Brianchon are stated and illustrated, and these are then applied to various problems in the construction of conics which are to satisfy certain initial conditions. The importance, and advantages of conic sections in aircraft design are pointed out, among these being increased production and interchangeability of parts resulting from mass-production methods.

Chapter 11 deals briefly with the analytic theory of conics, one application of which is the computation of "offsets" (ordinates) at various "stations" (abscissas) for lofting. Matching of conics—that is, the fitting together of two conics at a given point so that they will have a common tangent and the same curvature there—is discussed with some calculus. In this chapter as in the one preceding, results and methods are given for the most part without proofs.

The Appendix contains a summary of forty formulas from analytic geometry of which ten are for the rotation of various axes. There is also a 5-place table of the first four natural trigonometric functions for each minute.

Some features of the book might well be improved in a later revision. For instance, it seems to the reviewer that it would be highly desirable to include precise definitions of various aircraft terms (perhaps in the form of a glossary) such as root chord, tip chord, wing reference plane, common percent plane, *etc.*, which are, admittedly, shown on drawings. Although this is not a text in aeronautics, the lack of these definitions is confusing to the uninitiated reader because these terms are basic to many of the fundamental systems of axes discussed; it would perhaps not be too much to add that even the well initiated reader would derive some benefit from such definitions. Some explanations in the book are unnecessarily long; for example, the derivation of the expression for z_w on page 125 requires a quarter of the page, whereas it is an immediate consequence of eliminating y_w from the two equations immediately above. On page 127 there is the misleading statement that the equations of a line are three in number without mentioning the fact that only two are independent until the end of Example 4 on page 128.

As a whole, in spite of an opposite impression that the preceding paragraph might give, this book shows painstaking execution. Discussions are clear, and there is a copious supply of examples illustrating almost every section. These generally consist of some which are purely mathematical and others applied to aircraft. The reviewer checked through a few of the computations and found no errors. No actual misprints were detected. There are a few minor anomalies, such as the splitting of sin A at the end of line 13, page 66, and that of $y_1 \cos \beta$ near the bottom of page 135. There are many excellent drawings, which are neatly labeled and accurately done. The typography is all that could be desired. Exercises are provided in Chapters 1, 2, and 9.

This book should do much in promoting the use of analytic geometry in the more scientific development and construction of airplanes.

F. A. BUTTER, JR.

NEW BOOKS RECEIVED

College Algebra and Trigonometry. By F. H. Miller. New York, John Wiley and Sons, Inc.; London, Chapman and Hall, Ltd., 1945. 12+324 pages. $3.00.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to Howard Eves, College of Puget Sound, Tacoma 6, Washington

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 676. *Proposed by T. R. Running, University of Michigan*

Show that in using the Gregorian calendar the first day of a century cannot occur on a Sunday, a Wednesday, or a Friday.

E 677. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Let the altitudes of a tetrahedron $ABCD$ meet the circumsphere again in the points A', B', C', D'. Show that the volume of the solid $ABCDA'B'C'D'$ is three times that of the given tetrahedron.

E 678. *Proposed by H. D. Grossman, New York City*

Prove that the number of circular permutations of $n=\sum_{i=1}^{k} a_i$ objects of which a_1 are alike, a_2 are alike, *etc.*, is

$$\frac{1}{n}\sum_{d}\phi(d)\,\frac{(n/d)!}{(a_1/d)!(a_2/d)!\cdots(a_k/d)!},$$

where d ranges over all divisors of the g.c.d. of $a_1, a_2, \cdots, a_k$.

E 679. *Proposed by Marshall Naul, Cumberland, Md.*

Compute the length of the curve of intersection of the unit sphere around the origin with the right helicoid $z=\theta$.

E 680. *Proposed by Gordon Pall, McGill University*

Prove that a real determinant of order 6, with elements numerically not exceeding unity, cannot have a value greater than 160.

SOLUTIONS

A Duodenary Square Ending in Zero

E 642 [1944, 530]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In the duodenary scale, a certain number does not end in 0, but its square does. Show that the square ends in 30, and the cube in 60.

Solution by Monte Dernham, San Francisco. Excluding 0, we find 6 to be the only digit whose square is divisible by twelve. Working in the duodenary scale, we have

$$(10N+6)^2 = 100N^2 + 100N + 30,$$

$$(10N+6)^3 = 1000N^3 + 1600N^2 + 900N + 160.$$

Thus the square of every duodenary number ending in 6 terminates in 30, and the cube in 60.

Also solved by Murray Barbour, Colin Blyth, Leon Bunyan, D. H. Browne, W. E. Buker, Frank Hawthorne, Irving Kaplansky, Samuel Kramer, W. R. Ransom, E. D. Schell, E. P. Starke, Albert Wilansky, and the proposer.

Positive Terms of a Determinant

E 644 [1944, 530]. *Proposed by A. W. Goodman, Republic Aviation Corporation*

Do there exist determinants of order $n>2$, such that all the $n!$ terms of the expansion are positive?

Solution by Irving Kaplansky, Columbia University. No. Any 2×2 minor would necessarily have three terms of like sign, *viz.*, either three $+$'s and a $-$ or three $-$'s and a $+$. But it is already impossible to arrange a 3×2 matrix whose three 2×2 minors all have this structure.

Also solved by Murray Barbour, D. H. Browne, Howard Eves, Daniel Finkel, J. B. Kelly, E. D. Schell, Lowell Schoenfeld, E. P. Starke, and the proposer.

Editorial Note. This naturally raises the question, What is the maximum difference between the numbers of positive and negative terms which a determinant of order m can give on expansion? Or, equivalently, What is the largest value a determinant can have, if every element is either 1 or -1? We know that this value cannot exceed $m^{m/2}$, which is the value for the determinant of a "U-matrix" in the sense of Paley, "On orthogonal matrices," *Journal of Mathematics and Physics* (M. I. T.), vol. 12 (1933), pp. 311–320. (A U-matrix becomes an orthogonal matrix, in the strict sense, when all its elements are divided by $m^{1/2}$. The determinant is then ± 1. Hence before division the determinant was $\pm m^{m/2}$.) It can hardly be doubted that the value $m^{m/2}$ is actually attained whenever m is a multiple of 4. It certainly is when $m=4, 8, 12, \cdots, 88$, and in infinitely many higher cases; but a U-matrix of order 92 has not yet been constructed. Nor is it yet known how far the maximum value for a determinant of ± 1's can fall short of $m^{m/2}$ for intermediate values of m; *e.g.*, such a determinant of order $m=6$ cannot exceed 160, although $m^{m/2}=216$.

A Quadrangle with Given Sides and Area

E 643 [1944, 530]. *Proposed by W. E. Buker, Pittsburgh Public Schools*

The sides a, b, c, d of a plane quadrangle being given in order and also the area A, find the length of one of the diagonals.

Solution by H. N. Carleton, West Newbury, Massachusetts. In order to make the expression less cumbersome, we define

$$p = (d^2 + a^2)/4, \quad q = (c^2 + b^2)/4, \quad r = (d^2 - a^2)/4, \quad s = (c^2 - b^2)/4.$$

After some straightforward though voluminous work (which has been suppressed for the sake of brevity) we find that, if such a quadrangle exists, its diagonal from (a, b) to (c, d) may have either of the values

$$\sqrt{\frac{2\{(p+q)A^2+(p-q)(r^2-s^2)\} \pm 2A\sqrt{-A^4+2(2pq-r^2-s^2)A^2-2(p-q)(ps^2-qr^2)-(r^2-s^2)^2}}{A^2+(p-q)^2}}.$$

These are the positive roots of the equation

$$\{A^2 + (p - q)^2\}x^4 - 4\{(p + q)A^2 + (p - q)(r^2 - s^2)\}x^2 + 4\{A^4 + 2(r^2 + s^2)A^2 + (r^2 - s^2)^2\} = 0.$$

Conjugate Sections of a Sphere

E 646 [1944, 586]. *Proposed by Orrin Frink, Jr., Xenia, Ohio*

Prove that any two conjugate planes through a secant of a sphere meet the sphere in orthogonal circles.

Solution by J. H. Butchart, Grinnell College. Let the two conjugate planes p and q intersect in a line which meets the sphere in the point L. Let P be the pole of p. Since P is the vertex of a right circular cone whose base is the circle cut from the sphere by p, it is clear that LP is orthogonal to this circle at L. Since q passes through LP, which is a tangent to the sphere, LP is a tangent to the

circle cut by q from the sphere, and the circles cut by p and q are orthogonal.

Also solved by Howard Eves and J. M. Feld. For a proof using coordinates, see Graustein, *Introduction to Higher Geometry* (New York, 1940), p. 456, Theorem 8.

Barycentric Coordinates with Respect to a Regular Tetrahedron

E 647 [1944, 586]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Let (m_1, m_2, m_3, m_4) be the barycentric coordinates of a point G with respect to a regular tetrahedron $A_1A_2A_3A_4$ of edge a (so that G is the centroid of masses m_1, m_2, m_3, m_4 at A_1, A_2, A_3, A_4). Obtain an expression for the distance A_4G.

Solution by Howard Eves, Syracuse University. We shall employ the following theorem: For any point P,

$$\sum (m_i \cdot A_iP^2) = \sum (m_i \cdot A_iG^2) + \sum m_i \cdot GP^2. \tag{1}$$

(McClelland, *Geometry of the Circle*, Art. 55, gives a proof of this theorem for the plane. The same method can be used in three or more dimensions.) Now, if P coincides with A_1, A_2, A_3, A_4 in turn, the relation (1) yields

$$\begin{aligned}
a^2(s - m_1) &= (s + m_1)A_1G^2 + m_2A_2G^2 + m_3A_3G^2 + m_4A_4G^2, \\
a^2(s - m_2) &= m_1A_1G^2 + (s + m_2)A_2G^2 + m_3A_3G^2 + m_4A_4G^2, \\
a^2(s - m_3) &= m_1A_1G^2 + m_2A_2G^2 + (s + m_3)A_3G^2 + m_4A_4G^2, \\
a^2(s - m_4) &= m_1A_1G^2 + m_2A_2G^2 + m_3A_3G^2 + (s + m_4)A_4G^2,
\end{aligned}$$

where $s = m_1 + m_2 + m_3 + m_4$. Solving these equations for A_4G^2, we find

$$A_4G^2 = a^2(m_1^2 + m_2^2 + m_3^2 + m_2m_3 + m_3m_1 + m_1m_2)/(m_1 + m_2 + m_3 + m_4)^2.$$

This result is easily extended to the regular simplex in n-space.

Also solved by the proposer.

A Trigonometrical Determinant

E 648 [1944, 586]. *Proposed by Mary L. Boas and R. P. Boas, Jr., Tufts College and Harvard University*

Show that, when $x = 2\cos \pi/(n+1)$, the n-rowed determinant

$$\begin{vmatrix}
x & 1 & 0 & 0 \cdots 0 & 0 \\
1 & x & 1 & 0 \cdots 0 & 0 \\
0 & 1 & x & 1 \cdots 0 & 0 \\
0 & 0 & 1 & x \cdots 0 & 0 \\
\cdot & \cdot & \cdot & \cdot \cdots \cdot & \cdot \\
0 & 0 & 0 & 0 \cdots x & 1 \\
0 & 0 & 0 & 0 \cdots 1 & x
\end{vmatrix}$$

has the value zero.

Solution by J. B. Kelly, Langley Field, Va. Set $x=p+p^{-1}$, and let P_n denote the n-rowed determinant. Then

$$P_1 = x = p + p^{-1}, \qquad P_2 = x^2 - 1 = p^2 + 1 + p^{-2}.$$

Since $P_n = xP_{n-1} - P_{n-2}$ and $p^{n+1} - p^{-n-1} = (p+p^{-1})(p^n - p^{-n}) - (p^{n-1} - p^{-n+1})$, we can prove by induction that

$$P_n = p^n + p^{n-2} + \cdots + p^{-n+2} + p^{-n} = (p^{n+1} - p^{-n-1})/(p - p^{-1}).$$

Set $p = e^{\theta i}$, so that $x = 2\cos\theta$. Then

$$P_n = \frac{\sin(n+1)\theta}{\sin\theta},$$

which vanishes when

$$\theta = \frac{\pi}{n+1}.$$

Also solved by J. H. Butchart, Paul Civin, A. B. Farnell, N. J. Fine, Marguerite Z. and E. A. Hedberg, G. A. Hedlund, Irving Kaplansky, O. E. Lancaster, Leonard McFadden, M. F. Smiley, Alan Wayne, and the proposers.

Editorial Note. The following alternative solution is less significant, but rather entertaining. If $x = 2\cos\theta$, we have (identically)

$$\begin{aligned}
x\sin\theta - \sin 2\theta &= 0,\\
\sin\theta - x\sin 2\theta + \sin 3\theta &= 0,\\
-\sin 2\theta + x\sin 3\theta - \sin 4\theta &= 0,\\
&\cdots\cdots\cdots\\
\pm\sin(n-1)\theta \mp x\sin n\theta \pm \sin(n+1)\theta &= 0.
\end{aligned}$$

If $\theta = \pi/(n+1)$, the last term of the last equation vanishes, so we can formally eliminate $\sin\theta : \sin 2\theta : \cdots : \sin n\theta$ from the n equations (with that term omitted), obtaining the desired result immediately.

Hedlund remarks that

$$P_n = x^n - \binom{n-1}{1}x^{n-2} + \binom{n-2}{2}x^{n-4} - \cdots.$$

In other words, $P_n = U_n(x/2)$, where U_n is the Tschebyscheff polynomial of the second kind. (*Cf.* E 620 [1945, 44–46] and E 629 [1945, 100–101].) The determinantal expansion for $\sin(n+1)\theta$ was first given by Studnička in 1897.

An Almost Linear Set of Segments

E 649 [1944, 586]. *Proposed by L. A. Santaló, Rosario, Argentine Republic*

A set of parallel line segments will be called "linear" if all of them can be cut by one straight line. Show by an example that an infinite set of parallel segments in one plane may have the property that every subset of three is linear while the

whole set is not linear. (The segments are understood to be "open": not including their end points.)

I. *Solution by N. J. Fine, Lukas-Harold Laboratory, Indianapolis.* Let $f(x)$ be any real, positive-valued function such that $\lim_{|x|\to\infty} f(x) = 0$. Then the set of ordinates $\{0 < y < f(x)\}$ satisfies the required conditions. As a matter of fact, every finite subset is "linear."

A discrete set having the same property may be obtained by selecting those ordinates for which x is integral.

II. *Solution by Howard Eves, Syracuse University.* The segments

$$a_0, a_1, a_2, \cdots, \text{ and } b,$$

defined as follows, are seen to satisfy the requirements.

$$a_i\colon \quad x = 2^{-i}, \qquad 0 < y < 1;$$

$$b\colon \quad x = -1, \qquad 2 < y < 3.$$

Also solved by L. M. Kelly and the proposer. No *finite* set of segments can have this property; nor can any set of *intervals* (or "closed" segments). In other words, if every three of the intervals are linear, then the whole set is. See L. A. Santaló, "Complemento a la nota: sobre conjuntos de paralelepípedos de artistas paralelas," *Publicaciones del Instituto de Matemáticas*, vol. 3, no. 7 (1942); *Mathematical Reviews*, vol. 4 (1943), p. 112.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis 5, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4161. *Proposed by R. E. Gaines, University of Richmond*

Along a straight road a miles long there are n persons. What is the probability that no two persons are less than b miles apart?

4162. *Proposed by H. S. M. Coxeter, University of Toronto*

Prove, by the methods of real projective geometry, that if a projectivity $P \barwedge P'$ on a conic is not an involution, the envelope of PP' is a conic. (For a complex proof, see H. F. Baker, *Principles of Geometry*, Cambridge 1922 or 1930, p. 52.)

4163. *Proposed by W. H. Cullum, Washington Navy Yard*

In a vertical plane a heavy particle moves along a mathematical curve under the force of gravity, from a point A to a point B. The distance measured along the curve from A to B is the given length L. The initial velocity is V_0 while the velocity V at any point is such that $1/V-\lambda>0$, where λ is a positive quantity. Determine the time of passage from A to B such that it is a maximum.

4164. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Find a number of twelve digits $a_1a_2a_3a_4b_1b_2b_3b_4c_1c_2c_3c_4$ which is a perfect cube, and the cube root is the sum of the three numbers $a_1a_2a_3a_4$, $b_1b_2b_3b_4$, $c_1c_2c_3c_4$.

4165. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Let A and B be two fixed points of a given circle while C and D are two variable points of the same circle such that the arc length CD remains constant. The orthogonal projection of D on AC is P, of C on BD is Q. Show that, (1) The Simson lines of C and D for the triangles ABD and ABC have fixed directions. (2) The centers of the circles tritangent to DPQ (inscribed and escribed) describe two Pascal limaçons.

SOLUTIONS

Cube of Special Form

4114 [1944, 168]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

The digits 1, 2, 3, 4, 5, 6, 7, 8, 9 are taken each once and only once to form three numbers N, n, r so that $N=n^3-r$. Find these three numbers.

Solution by Marion L. Gaines, Student, University of North Carolina. Since every number is congruent to the sum of its digits mod 9, we have

$$n+r+N\equiv\sum_{v=1}^{9}v\equiv 0 \pmod 9.$$

On the other hand

$$N\equiv n^3-r \pmod 9.$$

Hence:

$$n+r+n^3-r\equiv n(n^2+1)\equiv 0 \pmod 9.$$

Since -1 is a quadratic non-residue mod 3, n^2+1 is not divisible by 3 and n must be divisible by 9.

If $n>100$, then $n^3=N+r>100^3$. Then n contains at least 3 digits and N or r at least 7 digits. This is impossible. Therefore we have to consider only the cases $n=9a$ where a is 1, 2, $\cdots$, 9.

The cases where $n=99$ or $n=90$ are impossible since n cannot contain either two 9's or a 0.

If $n=81$, then $n^3=531441$. Since n contains 2 digits, N and r must contain together 7 digits and one of them must contain exactly 6 digits in order that

the sum have 6 digits. The other number has only one digit and is therefore $\leqq 9$. The six digit number must contain the 1 at the fourth place from the right since by adding a number $\leqq 9$ we cannot raise the fourth place digit. This is impossible since 81 already contains the 1.

By the same reasoning the cases $n=72$, $n^3=373248$ and $n=54$, $n^3=157464$ are impossible since we would use 7 and 5 respectively twice.

For $n=36$, $n^3=46656$ it is evident that N and r contain 5 and 2 digits. Since the 6 has already been used we cannot obtain the 6 at the fourth place.

For $n=63$, $n^3=250047$. N or r must contain 0 at the fourth place. This is impossible.

For $n=45$, $n^3=91125$. N or r must have a 1 at the third and fourth place. This is impossible.

$n=9$ contains only one digit. Since $n^3=729$, the numbers N and r can have only 3 digits each. This is impossible.

For $n=18$, $n^3=5832$. We can use as last digits of N and r only the combinations 5, 7 and 3, 9. Since we have to carry over a 1 to the next place we can use only the same combinations in order to obtain this 3. Therefore at the last two places in both numbers, there are contained the odd numbers 3, 5, 7, 9. Since the 1 is already used we would have only even integers at the third place. Therefore we could not obtain the 8 since 1 has to be carried over.

Finally, we have to the case $n=27$, $n^3=19683$. It is easy to see that one of the numbers N or r must contain 5 digits, the other 2. The 5 digit number must start with 195 or 196. The 195 is impossible since none of the possible combinations gives us 3 for the last digit. For 196 we obtain the following four solutions:

19635 and 48; 19638 and 45; 19645 and 38; and 19648 and 35.

Solved also by Monte Dernham, W. W. Gandy, R. W. Hamming, Irving Kaplansky, J. B. Kelly, E. D. Schell, E. P. Starke, E. C. Varnum, R. H. Wilson, Jr., and the proposer.

A System of Tetrahedrons

4116 [1944, 234]. *Proposed by N. A. Court, University of Oklahoma*

Given the tetrahedron $(T_1)=SA_1B_1C_1$, the tangent place to its circumsphere at the diameter opposite of S meets the edges SA_1, SB_1, SC_1 in the points A_2, B_2, C_2. The tangent plane to the circumsphere of the tetrahedron $(T_2)=SA_2B_2C_2$ at the diametric opposite of S meets the edges of (T_2) through S in the points A_3, B_3, C_3 thus forming the tetrahedron $(T_3)=SA_3B_3C_3$, *etc.* Find the locus of the incenters of these tetrahedrons.

Solution by the Proposer. The tangent plane $A_2B_2C_2$ to the sphere $SA_1B_1C_1$ at the diametric opposite of S is parallel to the tangent plane to the same sphere at the point S, hence the plane $A_2B_2C_2$ is antiparallel to the plane $A_1B_1C_1$ for the trihedral angle $S-A_1B_1C_1$ (See the proposer's *Modern Pure Solid Geometry*, p. 247, art. 759). Similarly the tangent plane $A_3B_3C_3$ will be antiparallel to $A_2B_2C_2$ and therefore parallel to $A_1B_1C_1$.

Thus the tetrahedrons (T_1), (T_3), (T_5), $\cdots$ will be formed by a fixed trihedral angle $S-A_1B_1C_1$ and a series of parallel planes; hence these tetrahedrons will be homothetic. Consequently homologous points of these tetrahedrons, and in particular their incenters, will lie on a straight line passing through S.

Similarly for the tetrahedrons (T_2), (T_4), (T_6), $\cdots$.

Thus the required locus consists of two straight lines passing through the point S.

Note. The corresponding problem in the plane was considered in the *Journal de mathématiques élémentaires*, 1888, p. 233, Q. 250.

Editorial Note. This theorem may be extended to euclidean space of n dimensions. Let $T^i=(S, A_1^i, A_2^i, \cdots, A_n^i)$ be a non-degenerate simplex with the circumsphere (O^i) and diameter SO^iS^i; then $(SA_j^i)(SA_j^{i+1})=(SS^i)^2$, $(SA_j^{i+1})(SA^{i+2})=(SS^{i+1})^2$, and hence $(SA_j^i)/(SA_j^{i+2})=(SS^i)^2/(SS^{i+2})^1$, $1\leqq j\leqq n$. Therefore S is the center of similitude for T^i and T^{i+2}, and the theorem follows.

Commutative Transformations of a Polygon

4117 [1944, 234]. *Proposed by J. Rosenbaum, Bloomfield, Conn.*

A polygon $A_1A_2\cdots A_n$ may be transformed into a polygon $B_1B_2\cdots B_n$ by locating the points B_i on the sides A_iA_{i+1} so that the ratio of A_iB_i to B_iA_{i+1} is equal to a constant r. Prove that if T_1 and T_2 are two such transformations for the ratios r_1 and r_2, then $T_1\circ T_2=T_2\circ T_1$, and generalize.

Solution by R. C. Buck, Cambridge, Mass. Regard the vertices A_i as being determined by vectors, which need not lie in the same plane, or, in n-space, the same subspace. Then,

$$B_i=(1-r)A_i+rA_{i+1}=(1-r+rE)(A_i)$$

where $A_{n+1}=A_1$, and $E(A_i)=A_{i+1}$. Since E acts on the A_i only, and $T=1-r+rE$ is linear with *constant* coefficients, any collection of such operators commute. The same would be true for any transformation T of the form $T=P_0+P_1E+P_2E^2+\cdots+P_mE^m$, where the P_i are left unchanged by E and its powers.

Solved also by Howard Eves, Michael Goldberg, and the proposer.

Editorial Note. The solutions by Eves and Goldberg are similar and use vectors; Eves states in addition to the above extensions that $A_1, A_2, \cdots, A_n$ need not be points, but any elements (such as lines, circles, etc.) linear combinations of which yield the same kind of element. The proposer considered only plane polygons and used rectangular coordinates. The case was considered where the triangles $A_iA_{i+1}B_i$ are similar and constructed all exteriorly, or all interiorly, for a given transformation. In this case we may also use vectors or complex coordinates, but the expressions are much simpler for the latter, and the use of Buck's E adds additional simplicity. Thus $A_i=\rho_ie^{i\theta_i}$, $\alpha=$angle $A_{i+1}A_iB_i$, $r=A_iB_i/A_iA_{i+1}$, $\lambda=re^{i\alpha}$; then $B_i=[1+\lambda\Delta]A_i$, $\Delta A_i=A_{i+1}-A_i$, and $T=1+\lambda\Delta$. The desired result is obvious.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.

The Jagelonski Uniwersytet in Cracow will shortly resume a full schedule of activities under the rectorship of Professor Ler-Splawinski. Thirteen of the professors died in the concentration camp at Oranienburg and five others shortly after returning to Cracow; Antoni Hoberski is known to have perished. Even during the occupation secret work was furnished by the university to eight hundred students directed by over two hundred professors and scientists.

Professor R. P. Agnew of Cornell University has been awarded the degree of Doctor of Science by Allegheny College.

Dr. Mary D. Clement of Wells College has been promoted to an assistant professorship.

Professors G. C. Evans and Jerzy Neyman of the University of California are on leaves of absence for work with the War Department and the N.D.R.C. respectively.

Professor Tomlinson Fort of Lehigh University has been appointed professor of mathematics and head of the department at the University of Georgia.

Assistant Professor W. C. Janes of Kansas State College has been granted a leave of absence for research work with the Beachcraft Corporation of Wichita, Kansas.

Dr. Charles Loewner of Brown University has been appointed to an associate professorship at Syracuse University.

Professor L. J. Mordell of the University of Manchester has been elected Sadleirian Professor of Mathematics at Cambridge University, succeeding Professor G. H. Hardy.

S. T. Parker of the University of Louisville has been promoted to an assistant professorship.

Associate Professor George Polya of Stanford University has been appointed to a visiting professorship at the University of California.

Professor O. H. Rechard of the University of Wyoming has been appointed dean of the liberal arts college.

Assistant Professor C. E. Rhodes of Union College has been appointed professor of mathematics and head of the department at Washington College, Chestertown, Maryland.

G. F. Rose has been appointed by George Washington University to serve as research associate in the Allegheny Ballistics Laboratory, Cumberland, Maryland.

Dr. Stanislaw Ulam of the University of Wisconsin has been appointed to an associate professorship at the University of Southern California.

Professor Oscar Zariski of the Johns Hopkins University has been appointed to a visiting professorship for one year at the University of São Paulo, Brazil.

Dean Emeritus T. F. Holgate of Northwestern University died April 10, 1945. He was a charter member of the Association.

WAR INFORMATION

EDITED BY C. V. NEWSOM

Send news reports upon the utilization of mathematicians or mathematics in war activities to C. V. Newsom, Oberlin College, Oberlin, Ohio.

THE NATIONAL ROSTER

The National Roster of Scientific and Specialized Personnel is the division in the Bureau of Placement of the War Manpower Commission charged with the responsibility of obtaining the most effective utilization of professional and scientific personnel during the war period. When established in June, 1940, it was set up under the joint auspices of the United States Civil Service Commission and the National Resources Planning Board, because at that time it was believed that the Federal Government would be the principal user of the specialized personnel that would register with the Roster. It soon became apparent, however, that the need for these specialists would be greatest in industry and in the Armed Forces and, accordingly, when the President by Executive Order established the War Manpower Commission in April, 1942, he set forth in that order that the National Roster should be transferred from the Civil Service Commission to the War Manpower Commission. Dr. George A. Works is now Director of the National Roster; he replaced Dr. Leonard Carmichael on November 1, 1944. The address of the Roster is 1066 U Street, N.W., Washington 25, D. C.

The National Roster's principal task has been to register, recruit, and place professional and scientific personnel in those jobs in which they could best aid the war effort. At present, the Roster has registered and maintains in its records the education, experience, and training backgrounds of approximately half a million of the country's professionally qualified individuals. The data in these

records are kept current by the Roster's practice of recircularizing its registrants annually. The data obtained in this manner enable the Roster to carry on such important functions as: (1) the gathering and analysis of statistics to determine the characteristics and employment distribution of the various professions, (2) the making of studies of supply and demand in the important professions, (3) the publication of vocational guidance pamphlets in these professions, (4) the preparation of descriptions and definitions of the professions and their fields of specialization, (5) the provision of a free public placement service for the professional and scientific personnel of the country.

In performing its placement service, the National Roster carries on two types of activities: (1) the recruitment of personnel to fill specific jobs for which orders have been placed, and (2) positive placement designed to locate suitable positions for individuals who have indicated that they are or will soon become available for employment. The Roster receives job orders from private industry either directly or through the United States Employment Service, from federal agencies either directly or through the United States Civil Service Commission, from the Army and the Navy for military personnel usually to be considered for commissioning, from educational institutions, and from state and local governments. When a job order is received, a search is first made of papers maintained in a special file in which are contained brief statements of qualifications of individuals who have indicated that they are actively seeking employment or a change in employment. If the order can not be filled from this source, a machine run is made from the Roster's punch-card records, and qualified registrants are selected. The detailed records of these registrants are then examined to make certain that they possess the necessary qualifications and also to determine if the transfer would be in the interest of the war effort. As a result of this examination, a number of registrants are tentatively selected for referral. In most cases, these registrants will be employed in war jobs and it is important to determine their availability for transfer. The Roster's procedure is to write the registrant's employer, indicate the nature of the war job which has to be filled, and ask the employer whether he can grant his employee a release. Most employers are cooperative and if they believe the job to be filled is more important to the war effort than the job the employee is performing, they will agree to release him. The employer will give to his employee a form sent to him by the Roster on which there will be a brief description of the job in question and the employee will be asked to indicate on this form whether he is interested in being considered for transfer. In the event he notifies the Roster of his willingness to be considered for the position, his entire record is then referred to the establishment which has placed the job order with the Roster. This establishment will then deal directly with the registrant. Should the employer be a federal agency, the papers will be referred through the United States Civil Service Commission; referrals to industrial establishments are made through the United States Employment Service; military assignments are referred to the Army and Navy; and negotiations with colleges and universities are made directly.

Professionally qualified persons may write the Roster that they are seeking employment or they may register with the nearest local office of the United States Employment Service which will in turn notify the Roster. The Roster and the United States Employment Service have developed working relationships which make it possible for the local office to quickly refer job orders and job applicants to the Roster. Upon receipt of such notice, all current job orders in the Roster are checked, and if the individual is qualified, his papers are immediately referred for consideration. If there is no current job order for which he is qualified, a search is made through prior orders and other information which the Roster has on hand to locate establishments where individuals with such qualifications have been employed. When these have been located, a record of the registrant's qualifications is sent to such employers for consideration. In effect, the Roster solicits jobs for these people. If neither of the above steps produces results, a brief record of the individual's qualifications is transmitted to a few selected local United States Employment Service offices which are requested to inquire among employers in their respective areas for possible employment.

Professionally qualified members of the Armed Forces who are about to be discharged are invited to avail themselves of the Roster's services. At present, the Roster has a number of standing orders from large industrial concerns for engineers, chemists, and physicists. Orders have and are now being received specifying a desire for veterans. These positions offer opportunities for war work now and for permanent employment in the postwar period.

REGISTRATION WITH THE ROSTER ON DECEMBER 31, 1944

	Registration			*Median Age*	*Education*	
	Total	*Male*	*Female*		*Doctors*	*Masters*
Total Registration	439,757	420,120	19,637	37.9	37,434	59,918
Mathematics	14,610	12,699	1,911	36.6	1,826	5,188
Actuarial Science	882	848	34	41.0	23	188
Statistics	2,435	1,978	457	37.2	282	634

AMERICAN-SOVIET SCIENCE COMMITTEE

The Science Panel of the Congress celebrating the tenth anniversary of U. S. recognition of the Soviet Union was held on November 7, 1943. Shortly thereafter a permanent Science Committee, affiliated with the National Council of American-Soviet Friendship, was formed with the purpose of facilitating scientific exchange in both directions between the U.S.A. and the U.S.S.R. The purposes of the permanent Committee were laid down as follows: "The primary purpose of the Science Committee of the National Council of American-Soviet Friendship is to bring the work in natural science in these two countries into closer and more continuous contact. It proposes to accomplish this purpose by facilitating the exchange between the U.S.A. and the Soviet Union of scientific

publications, of scientific personnel, and of students of natural science, and by acting as a clearing center for information from, and about science in the U.S.S.R. For the present, it proposes to restrict its activities to the natural sciences—astronomy, mathematics, physics, chemistry, biology, geology, and the basic sciences on which medicine, agriculture, and technology rest. Its most pressing immediate problem is the exchange and diffusion of scientific knowledge between the U.S.A. and the U.S.S.R., chiefly through publications and through exchange visits of scientists, both during and after the war." A large number of American scientists, including mathematicians, are members of the Science Committee. Headquarters of the Organization are located at 232 Madison Ave., New York 16, N. Y.

The Committee recently started the publication of a bulletin which may be of interest to many scientists. A feature of each number is a bibliography of books and manuscripts recently received from the U.S.S.R. The following quotations are from the first two issues of the bulletin.

"The Science Section of the USSR Society for Cultural Relations with Foreign Countries, generally known as VOKS, regularly communicates with our Committee and has sent us scientific journals and books, including those issued in 1944, together with scientific manuscripts which we are asked to prepare for publication in the U.S.A. We also receive their valuable periodical, *Science in the USSR*, which is published in English and contains scientific news and reports of a general nature. Scientific material received by the U. S. Department of State and by the Soviet Embassy is also sent to this Committee for abstracting, review, and distribution to American scientists. In return, reprints and gifts of books and periodicals for libraries damaged by the war are being sent from the U.S.A. to the Soviet Union."

"Any American scientist, interested in the work which his Soviet colleagues are now carrying on, is requested to write for information to the Science Committee. Members have the privilege of requesting specific material from the Soviet Union. A temporary repository for a number of the books, journals, and papers from the Soviet Union, has been arranged at the American-Russian Institute, 58 Park Avenue, New York City, for the convenience of Committee members. Material may be borrowed for a limited, but sufficient length of time, in return for a small fee to cover mailing charges. Requests should be addressed to the Science Committee."

"An arrangement has been made by which the Science Committee will turn over to *Mathematical Reviews* one copy of each mathematical journal received from the Soviet Union. Each paper will be reviewed promptly in *Mathematical Reviews*. A microfilm copy of the Russian original of each paper, together with the journal, will be deposited in the Brown University Library, which will furnish microfilm copies at the usual rates. Journals such as *Doklady*, which contain articles in other fields as well as mathematics, and physical and astronomical journals will also be sent to *Mathematical Reviews*, which will review those papers of interest to mathematicians. The tables of contents of such

'mixed' journals will be published in the Bulletin of the Science Committee. The journals will then be deposited with the Brown University Library, which, upon order, will furnish a microfilm or photostat copy of any paper at the usual rates. Correspondence concerning all papers so listed may be addressed to the Librarian, Brown University, Providence, R. I."

"A catalogue from the Soviet Union announces a new newspaper and magazine subscription service conducted by Kniga, Moscow. . . . Besides journals in economics, sociology, and political science, there are listed 11 in the physical sciences and mathematics; . . . . Two scientific publications contain articles in English, French, or German: *Acta Physiocochimica URSS* and *Journal of Physics*. Catalogues, with subscription prices, may be obtained from the Four Continent Book Corporation, 253 Fifth Avenue, New York City. Due to a United States post office ruling, Four Continents is, however, unable to enter any subscriptions to Russian journals. Persons wishing to subscribe must enter their own subscriptions by writing directly to Kniga in Moscow, enclosing a bank draft covering subscription costs."

CONGRESSIONAL STUDY OF HIGHER EDUCATION

House Resolution 63 of the 79th Congress authorized the appointment of an Advisory Committee of educators to conduct a study of the effect of the war upon higher education. This Committee recently made its report. Among the specific recommendations contained in the report are the following.

"The Committee recognizes the present serious shortage of manpower to meet the demands of war. It specifically calls attention, however, to the rapidly increasing shortage of men in professional fields essential to the national welfare, such as, medicine, dentistry, engineering, physics, chemistry, divinity and others. With each year of war the gap in the flow of young men into these essential fields becomes a more serious threat to the national health, safety or interest. The Committee recommends that: *A*. At the earliest possible time Selective Service reestablish student deferment, for those majoring in fields essential to the national welfare and for which extended periods of training are necessary. *B*. In plans for National Service, provision be made in the legislation for the exemption, for the period of their training, of students for fields essential to the national welfare and for which extended periods of training are necessary.

"The War Department has announced its plan of a point system as a basis for discharge of military personnel. In the plan as stated, no provision was made for preferential discharge of former students preparing for fields essential to the national welfare. Since such men have partly completed their education, preferential discharge will lessen the gap in the flow of trained men and women into such fields. The Committee recommends that: The War and Navy Departments include in their bases for discharge, consideration of the educational plans of those who have completed two or more years of college education in essential fields and who will continue their education in these fields after their discharge.

"It is essential in the national interest that higher educational institutions

continue to function at a high level of effectiveness. This can not be done if faculty members in areas of teacher shortage are inducted into the armed forces or national service. The Committee recommends that: *A*. Faculty members teaching in essential fields be deferred to meet the educational needs of veterans and others. *B*. In plans for National Service faculty members of higher educational institutions, teaching in necessary fields, be considered as being engaged in an essential activity.

"In certain fields of instruction in higher educational institutions there is now a serious shortage in teaching personnel. The Committee recommends that: Members of faculties of higher educational institutions whose services are requested by the institutions be given priority in release from military duty or other government positions.

"The war demonstrated the effectiveness of national coordination and cooperation in carrying forward research essential in the national welfare. The gains made through the successful operation of the Office of Scientific Research and Development should be retained and should be broadened to include research in the social sciences and humanities. Higher educational institutions are the principal agencies for the training of scientific and research workers of the future. It is therefore essential that as much scientific work as possible be widely distributed among the higher educational institutions in order (*a*) To accomplish definite objectives in specific instances of research; and, (*b*) To utilize effectively the staff at the institutions so as to extend and continue the effective training of a competent corps of research workers. The administration of such research programs might well be through a special agency operating somewhat as the existing Office of Scientific Research and Development. The Committee recommends that: A federal research agency be established or designated by the Congress and directed to use, on a contractual basis, higher educational institutions for developing and conducting of research and the training of research workers. Funds should be appropriated for such research to be used for specific programs when approved by the research agency.

"When World War II was declared, no plan had been prepared for the effective utilization of institutions of higher education. To avoid the repetition of such lack of planning, and in order that the universities and colleges may be utilized most effectively in the national interest during a period of declared emergency, the Committee recommends that: A committee representing the educational institutions and the armed services prepare a unified plan, which should be revised periodically, for using the colleges and universities in declared national emergencies."

THE MATHEMATICAL ASSOCIATION OF AMERICA

THE ANNUAL MEETING OF THE NORTHERN CALIFORNIA SECTION

The seventh annual meeting of the Northern California Section of the Mathematical Association of America was held at the University of San Francisco on Saturday, January 27, 1945. The Chairman of the Section, Professor Gabor Szegö, presided during the first part of the morning session, and Professor R. M. Robinson presided for the remainder of the morning session. Mr. S. A. Francis presided at the afternoon session.

The attendance was sixty, including the following twenty-two members of the Association: H. M. Bacon, G. A. Baker, H. W. Becker, S. A. Francis, F. T. Frank, Emma V. Hesse, D. H. Lehmer, Sophia Levy McDonald, E. D. Miller, F. R. Morris, W. H. Myers, George Polya, R. M. Robinson, S. A. Schaaf, Ethel Spearman, Pauline Sperry, Ruth G. Sumner, Gabor Szegö, K. J. Waider, R. K. Wakerling, Harriet A. Welch, A. R. Williams.

The following officers were elected for the coming year: Chairman, Pauline Sperry, University of California; Vice-Chairman, W. H. Myers, San Jose State College; Secretary-Treasurer, H. M. Bacon, Stanford University; representative on the *California Journal of Secondary Education*, Ruth G. Sumner, Oakland High School.

By invitation of the Section, Professor A. C. Schaeffer of Stanford University gave an hour's address during the morning session.

The following papers were presented:

1. *An application of the conchoid of Nicomedes*, by Professor F. R. Morris, Fresno State College.

It was pointed out that the problem of deciding whether a board can be moved from one corridor into another at right angles to the first is, under certain assumptions, equivalent to that of finding the maximum distance of a point on the loop of the conchoid from the axis of the loop. This principle was used to illustrate three methods of solving the ladder problem.

2. *Heat conduction problems and the Laplace transform*, by Dr. S. A. Schaaf, University of California.

The speaker stated that in recent years Carslaw, Churchill, and others have developed a method for solving boundary value problems by use of the Laplace transform. The method is particularly well adapted to problems in heat conduction, where its application has yielded solutions of many difficult problems. The method was described in general terms, and several simple illustrative examples were given.

3. *Differentiable functions*, by Professor A. C. Schaeffer, Stanford University, introduced by the Secretary.

Professor Schaeffer gave an elementary proof of a classical theorem of Serge Bernstein which states that if in an interval $a \leqq x < b$ all the derivatives of a function $f(x)$ exist and are non-negative, then the function is represented by its Taylor's expansion in powers of $x-a$ in the interval $a \leqq x < b$. This theorem fills a definite need in elementary calculus courses, and its proof is not beyond the range of beginning students. It is customary in introductory calculus courses, when expanding $\tan x$ or $\sec x$ in Maclaurin's series, to find only the first few terms, and omit any discussion of convergence. The theorem of Bernstein reveals that the Maclaurin series for $\tan x$ converges to $\tan x$ when $0 \leqq x < \pi/2$. It is then quite easy to prove that the series represents $\tan x$ when x lies between $-\pi/2$ and zero. A similar procedure may be used to find the interval of convergence of the Maclaurin's series for $\sec x$, or of the Taylor's series in powers of $x-(\pi/2)$ for $\cot x$ and $\csc x$. The theorem of Bernstein has motivated a considerable amount of interesting research in recent years. There are several closely related problems which are still open questions.

4. *Combinatory analysis in electronics*, by H. W. Becker, Mare Island Fire Control School, United States Naval Training School.

The speaker remarked that electronic components may be represented by two types of symbols, namely, the schematic and the mathematical. The schematic diagram is best for the mechanic or student. But by assigning mathematical symbols to components as operands and connections as operators, one can represent and solve a circuit in the same line of type. This is the approach of Riordan, Shannon, Kron, and Becker. The procedure can be applied to pararithmetic, network severances, selector switch algebra, splices in Latin space, Bell's numbers, and operational equations.

5. *Constructions of triangles under certain conditions*, by Professor A. R. Williams, University of California.

It was explained that the title of this paper was intentionally non-committal. Professor Williams remarked that a triangle with given sides may be constructed on the sphere by use of a graduated great circle, and a triangle with given angles by use of the polar triangle. The principal purpose of the paper was to give some idea of the value of the study of non-euclidean geometry. The construction of a triangle on the hyperbolic plane when three angles (whose sum is less than two right angles) are given was explained. The construction of Liebmann by use of a series of related right triangles was given.

6. *Teacher's organizations, their place in the school program*, by Mrs. Ruth G. Sumner, Oakland High School.

Mrs. Sumner spoke of the various organizations whose membership is open to teachers, and called attention to those from whose activities the teacher of mathematics receives impetus for professional growth. She described briefly the

studies which are being carried out by some of the mathematical associations, and pointed out the desirability of coordinating the efforts of all the committees now studying problems which are more or less identical.

7. *Recursive functions*, by Professor R. M. Robinson, University of California.

The speaker considered only functions with natural numbers for arguments and functional values. A recursive function was defined to be one which can be obtained from the constant, identity, and successor functions by repeated substitutions and recursions. A typical recursion schema is

$$F(x, 0) = G(x), \qquad F(x, y + 1) = H[x, y, F(x, y)],$$

which defines $F(x, y)$ in terms of $G(x)$ and $H(x, y, z)$. It was shown that $x+y$, xy, x^y, $x!$, $|x-y|$, and $[y/x]$ are recursive functions. For example, the recursive definition of x^y is $x^0=1$, $x^{y+1}=x^y \cdot x$. It was shown that to define $[y/x]$ we may take

$$G(x) = 0, \qquad H(x, y, z) = z + 0^{|(y+1)-x(z+1)|}.$$

H. M. BACON, *Secretary*

CALENDAR OF FUTURE MEETINGS

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS
INDIANA, Indianapolis, October 19, 1945
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MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA
NORTHERN CALIFORNIA, Berkeley January 26, 1946
OHIO
OKLAHOMA
PHILADELPHIA, Philadelphia, December 1, 1945
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